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MATH PROGRAM

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CHAPTER (I)

THE DERIVATIVE

2.1 TANGENT LINES AND RATES OF CHANGE :

Tangent Lines :

* Referring to Figure 2.1.1 the slope m_{PQ} of the secant line through $P(x_0, f(x_0))$ and $Q(x, f(x))$ on the curve of $y = f(x)$ is

$$m_{PQ} = \frac{f(x) - f(x_0)}{x - x_0}.$$

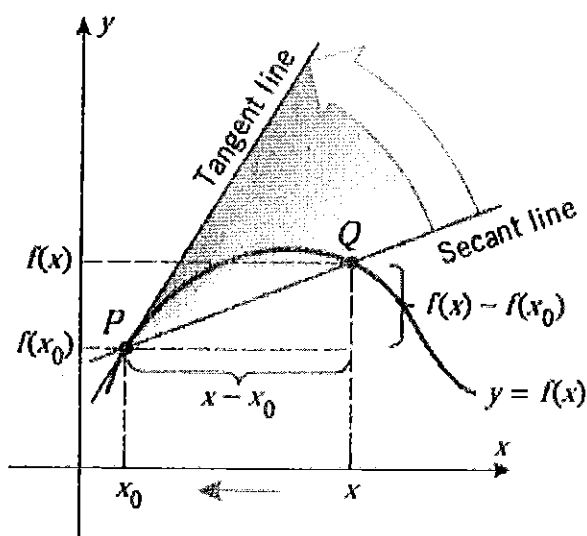


Figure 2.1.1

* If we let x approach x_0 , then the point Q will move along the curve and approach the point P . If the secant line through P and Q approaches a limiting position as $x \rightarrow x_0$, then we will regard that position to be the position of the tangent line at P .

2.1.1 Definition :

Suppose that x_0 is in the domain of the function f . The tangent line to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the line with equation

$$y - f(x_0) = m_{tan}(x - x_0)$$

where

$$m_{tan} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad (1)$$

provided the limit exists. For simplicity, we will also call this the tangent line to $y = f(x)$ at x_0 .

Example 1 :

Use definition 2.1.1 to find an equation for the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution

* Applying Formula (1) with $f(x) = x^2$ and $x_0 = 1$, we have

$$\begin{aligned} m_{tan} &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \end{aligned}$$

Remember that :

$$* (a^2 - b^2) = (a - b)(a + b)$$

$$= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} (x + 1) = \boxed{2}.$$

* Thus , the tangent line to $y = x^2$ at $(1,1)$ has equation

$$y - f(x_0) = m_{tan} (x - x_0)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

or equivalently

$$\boxed{y = 2x - 1}.$$

* There is an alternative way of expressing Formula (1) that is commonly used. If we let h denote the difference

$$h = x - x_0$$

then the statement that $x \rightarrow x_0$ is equivalent to the statement $h \rightarrow 0$, so we can rewrite (1) in terms of x_0 and h as

$$\boxed{m_{tan} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}} \quad (2)$$

* Figure 2.1.2 shows how Formula (2) express the slope of the tangent line as a limit of slopes of secant lines.

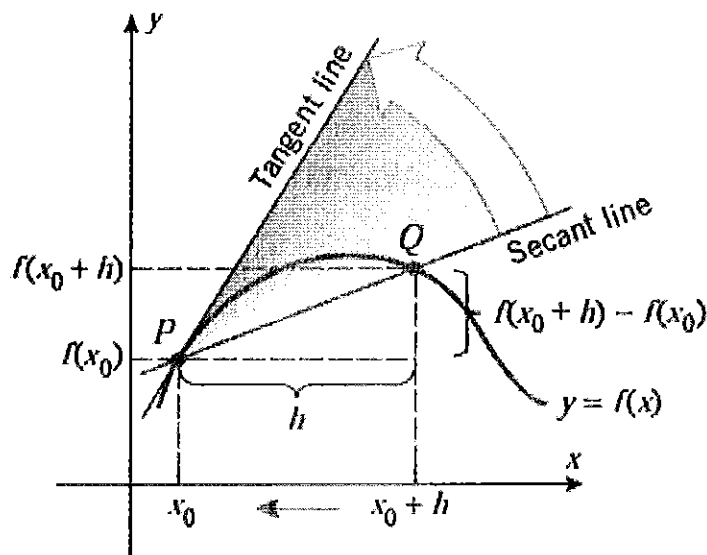


Figure 2.1.2

Example 3 :

Find an equation for the tangent line to the curve $y = 2 / x$ at the point $(2, 1)$ on this curve.

Solution

*** First , we will find the slope of the tangent line by Formula (2) with $f(x) = 2 / x$ and $x_0 = 2$. This yields**

$$\begin{aligned} m_{tan} &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{2}{2+h} - 1}{h} \end{aligned}$$

Remember that :

$$* \frac{a}{b} \pm 1 = \frac{a \pm b}{b}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{\left(\frac{2 - (2 + h)}{2 + h} \right)}{h} = \lim_{h \rightarrow 0} \frac{-h}{h(2 + h)} \\ &= - \left(\lim_{h \rightarrow 0} \frac{1}{2 + h} \right) = \boxed{-\frac{1}{2}}. \end{aligned}$$

*** Thus , an equation of the tangent line to $y = 2 / x$ at $(2, 1)$ is**

$$y - f(x_0) = m_{tan}(x - x_0)$$

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y - 1 = -\frac{1}{2}x + 1$$

or equivalently

$$y = -\frac{1}{2}x + 2.$$

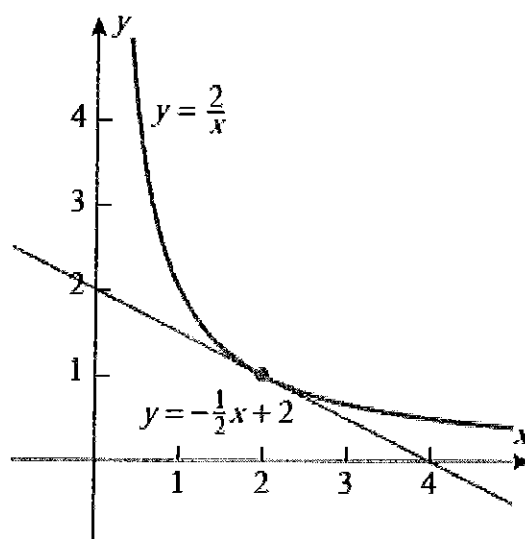


Figure 2.1.3

Example 4 :

Find the slopes of the tangent lines to the curve $y = \sqrt{x}$ at $x_0 = 1$, $x_0 = 4$, and $x_0 = 9$.

Solution

* We could compute each of these slopes separately, but it will be more efficient to find the slope for a general value of x_0 and substitute the specific numerical values. Proceeding in this way we obtain

$$\begin{aligned} m_{tan} &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \cdot \frac{\sqrt{x_0 + h} + \sqrt{x_0}}{\sqrt{x_0 + h} + \sqrt{x_0}} \end{aligned}$$

Remember that :

$$* (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$$

$$= \lim_{h \rightarrow 0} \frac{x_0 + h - x_0}{h(\sqrt{x_0 + h} + \sqrt{x_0})}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x_0 + h} + \sqrt{x_0})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x_0 + h} + \sqrt{x_0}} = \boxed{\frac{1}{2\sqrt{x_0}}}.$$

* The slopes at $x_0 = 1$, 4 , and 9 can now be obtained by substituting these values into our general formula for $m_{\tan}$.
Thus,

$$\text{slope at } x_0 = 1 : \frac{1}{2\sqrt{1}} = \boxed{\frac{1}{2}}.$$

$$\text{slope at } x_0 = 4 : \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}.$$

$$\text{slope at } x_0 = 9 : \frac{1}{2\sqrt{9}} = \boxed{\frac{1}{6}}.$$

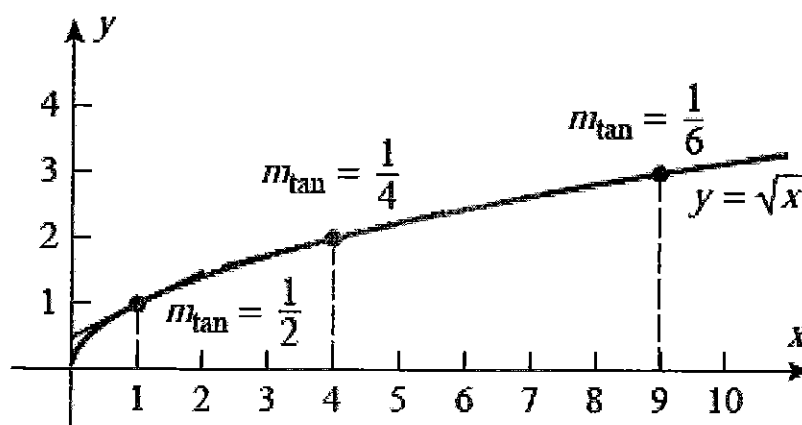


Figure 2.1.4

EXERCISE SET 2.1: (Home Work)

15-18 A function $y = f(x)$ and an x -value x_0 are given.

(a) Find a formula for the slope of the tangent line to the graph of f at a general point $x = x_0$.

(b) Use the formula obtained in part (a) to find the slope of the tangent line for the given value of x_0 .

15. $f(x) = x^2 - 1$; $x_0 = -1$

16. $f(x) = x^2 + 3x + 2$; $x_0 = 2$

17. $f(x) = x + \sqrt{x}$; $x_0 = 1$

18. $f(x) = 1/\sqrt{x}$; $x_0 = 4$

2.2 THE DERIVATIVE FUNCTION :

Definition of the Derivative Function :

* In the last section we showed that if the limit

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists, then it can be interpreted as the slope of the tangent line m_{tan} to the curve $y = f(x)$ at $x = x_0$.

* This limit is so important that it has a special notation :

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}. \quad (1)$$

2.2.1 Definition :

The function f' defined by the formula

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \quad (2)$$

is called the derivative of f with respect to x . The domain of f' consists of all x in the domain of f for which the limit exists.

Example 1 :

Find the derivative with respect to x of $f(x) = x^2$, and use it to find the equation of the tangent line to $y = x^2$ at $x = 2$.

Solution

* It follows from (2) that

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

Remember that :

$$* (a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = \lim_{h \rightarrow 0} (2x+h) = \boxed{2x}.$$

* Thus , the slope of the tangent line to $y = x^2$ at $x = 2$ is $f'(2) = 2(2) = 4$.

* Since $y = (2)^2 = 4$ if $x = 2$, thus an equation of the tangent line to $y = x^2$ at $(2, 4)$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y - 4 = 4(x - 2)$$

$$y - 4 = 4x - 8$$

or equivalently

$$\boxed{y = 4x - 4}.$$

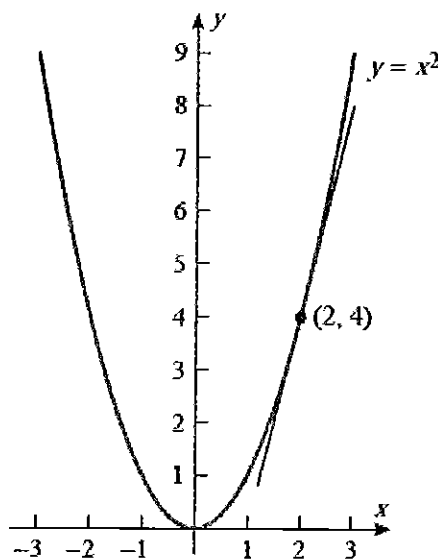


Figure 2.2.1

Finding an Equation for the Tangent Line to $y = f(x)$ at $x = x_0$:

Step 1. Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

Step 2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.

Step 3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or, equivalently,

$$y = f(x_0) + f'(x_0)(x - x_0) \quad (3)$$

Example 2 :

(a) Find the derivative with respect to x of $f(x) = x^3 - x$.

Solution

* It follows from (2) that

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \end{aligned}$$

Remember that :

$$* (a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

$$= \lim_{h \rightarrow 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h[3x^2 + 3xh + h^2 - 1]}{h}$$

$$= \lim_{h \rightarrow 0} [3x^2 + 3xh + h^2 - 1] = \boxed{3x^2 - 1}.$$

Other Derivative Notations :

- * When the independent variable is x , the differentiation operation is also commonly denoted by

$$f'(x) = \frac{d}{dx}[f(x)] \quad \text{or} \quad f'(x) = D_x[f(x)].$$

- * In the case where there is a dependent variable $y = f(x)$, the derivative is also commonly denoted by

$$f'(x) = y'(x) \quad \text{or} \quad f'(x) = \frac{dy}{dx}.$$

- * With the above notations , the value of the derivative at a point x_0 can be expressed as

$$f'(x_0) = \frac{d}{dx}[f(x)]\bigg|_{x=x_0}, \quad f'(x_0) = D_x[f(x)]\bigg|_{x=x_0},$$

$$f'(x_0) = y'(x_0), \quad f'(x_0) = \frac{dy}{dx}\bigg|_{x=x_0}$$

- * It is common to regard the variable h in the derivative formula

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (9)$$

as an increment Δx in x and write (9) as

$$\boxed{f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}} \quad (10)$$

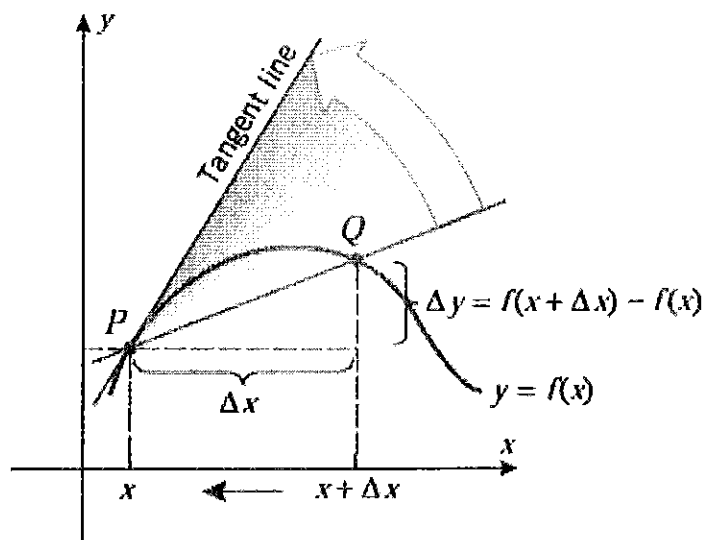
- * Moreover , if $y = f(x)$, then the numerator in (10) can be regarded as the increment

$$\Delta y = f(x + \Delta x) - f(x) \quad (11)$$

in which case

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (12)$$

* The geometric interpretation of Δx and Δy are shown in Figure 2.2.14.



$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

Figure 2.2.14

EXERCISE SET 2.2: (Home Work)

9,13 Use Definition 2.2.1 to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = a$.

9. $f(x) = 2x^2$; $a = 1$

13. $f(x) = \sqrt{x+1}$; $a = 8$

15,17 Use Formula (12) to find dy/dx .

15. $y = \frac{1}{x}$

17. $y = x^2 - x$

2.3 INTRODUCTION TO TECHNIQUES OF DIFFERENTIATION :

Derivative of a Constant :

* The simplest kind of function is a constant function $f(x) = c$. Since the graph of f is a horizontal line of slope 0, the tangent line to the graph of f has slope 0 for every x ; and hence we can see geometrically that $f'(x) = 0$ (Figure 2.3.1).

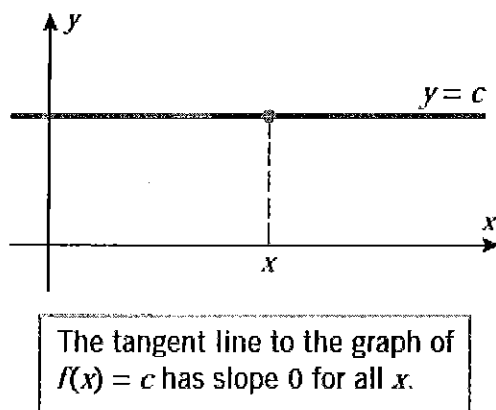


Figure 2.3.1

* We can also see this algebraically since

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0 \end{aligned}$$

* Thus, we have establish the following result.

2.3.1 Theorem :

The derivative of a constant function is 0 ; if c is any real number , then

$$\boxed{\frac{d}{dx}[c] = 0} . \quad (1)$$

Example 1 :

- * $\frac{d}{dx}[1] = \boxed{0} .$
- * $\frac{d}{dx}[-3] = \boxed{0} .$
- * $\frac{d}{dx}[\pi] = \boxed{0} .$
- * $\frac{d}{dx}[-\sqrt{2}] = \boxed{0} .$

Remember that :

$$* \frac{d}{dx}[c] = 0$$

Derivative of Power Functions :

* *The simplest power function is $f(x) = x$. Since the graph of f is a line of slope 1, it follows that $f'(x) = 1$ for all x (Figure 2.3.2).*

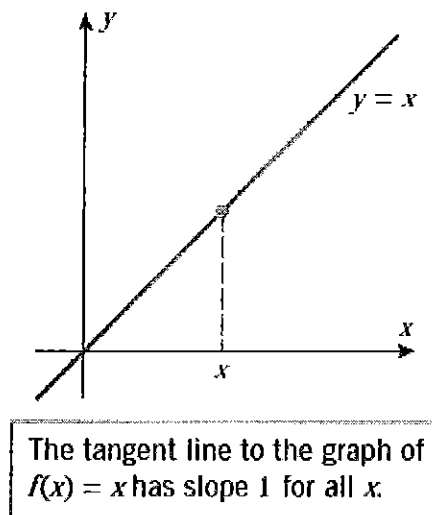


Figure 2.3.2

* *In other words,*

$$\boxed{\frac{d}{dx}[x] = 1} \quad (2)$$

* *Example 1 of Section 2.2 shows that the power function $f(x) = x^2$ has derivative $f'(x) = 2x$. From Example 2 in that section one can infer that the power function $f(x) = x^3$ has derivative $f'(x) = 3x^2$. That is,*

$$\frac{d}{dx}[x^2] = 2x \quad \text{and} \quad \frac{d}{dx}[x^3] = 3x^2 \quad (3-4)$$

* *These result are special cases of the following more general result.*

2.3.2 Theorem (The Power Rule):

If n is a positive integer, then

$$\frac{d}{dx}[x^n] = n x^{n-1} \quad (5)$$

Example 2 :

$$* \frac{d}{dx}[x^4] = 4x^3.$$

$$* \frac{d}{dx}[x^5] = 5x^4.$$

$$* \frac{d}{dt}[t^{12}] = 12t^{11}.$$

Remember that :

$$* \frac{d}{dx}[x^n] = n x^{n-1}$$

** Although our proof of the power rule in Formula (5) applies only to positive integer powers of x , it is not difficult to show that the same formula holds for all integer powers of x . Also saw in Example 4 of Section 2.2 that*

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \quad (6)$$

which can be expressed as

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2} x^{(1/2)-1} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

2.3.3 Theorem (Extended Power Rule):

If r is any real number, then

$$\frac{d}{dx} [x^r] = r x^{r-1}. \quad (7)$$

In words, to differentiate a power function, decrease the constant exponent by one and multiply the resulting power function by the original exponent.

Example 3 :

$$* \frac{d}{dx} [x^\pi] = \pi x^{\pi-1}.$$

$$* \frac{d}{dx} \left[\frac{1}{x} \right] = \frac{d}{dx} [x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$

$$* \frac{d}{dw} \left[\frac{1}{w^{100}} \right] = \frac{d}{dw} [w^{-100}] = -100 w^{-101} = -\frac{100}{w^{101}}.$$

$$* \frac{d}{dx} [x^{4/5}] = \frac{4}{5} x^{(4/5)-1} = \frac{4}{5} x^{-1/5} = \frac{4}{5 x^{1/5}} = \frac{4}{5 \sqrt[5]{x}}.$$

$$* \frac{d}{dx} [\sqrt{x}] = \frac{d}{dx} [x^{1/2}] = \frac{1}{2} x^{-1/2} = \frac{1}{2 \sqrt{x}}.$$

$$* \frac{d}{dx} [\sqrt[3]{x}] = \frac{d}{dx} [x^{1/3}] = \frac{1}{3} x^{-2/3} = \frac{1}{3 x^{2/3}} = \frac{1}{3 \sqrt[3]{x^2}}.$$

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}, \quad \frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$$

Derivative of a Constant Times a Function :

2.3.4 Theorem (Constant Multiple Rule):

If f is differentiable at x and c is any real number, then cf is also differentiable at x and

$$\boxed{\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)]} \quad (8)$$

Note :

* Formula (8) can also be expressed in function notation as

$$\boxed{(cf)' = cf'}$$

In words, a constant factor can be moved through a derivative sign.

Example 4 :

$$* \frac{d}{dx}[4x^8] = 4 \frac{d}{dx}[x^8] = 4[8x^7] = \boxed{32x^7}.$$

$$* \frac{d}{dx}[-x^{12}] = (-1) \frac{d}{dx}[x^{12}] = \boxed{-12x^{11}}.$$

$$* \frac{d}{dx}\left[\frac{\pi}{x}\right] = \pi \frac{d}{dx}[x^{-1}] = \pi[-x^{-2}] = \boxed{-\frac{\pi}{x^2}}.$$

Remember that :

$$* \frac{d}{dx}[c f(x)] = c \frac{d}{dx}[f(x)]$$

Derivatives of Sums and Differences :

2.3.5 Theorem (Sum and Difference Rules):

If f and g are differentiable at x , then so are $f + g$ and $f - g$ and

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \quad (9)$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \quad (10)$$

** Formula (10) can be proved in a similar manner or, alternatively, by writing $f(x) - g(x)$ as $f(x) + (-1)g(x)$ and then applying Formula (8) and (9).*

In words, the derivative of a sum equals the sum of the derivatives, and the derivative of a difference equals the difference of the derivatives.

Example 5 :

$$* \frac{d}{dx}[2x^6 + x^{-9}] = \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}]$$

$$= 2(6)x^5 + (-9)x^{-10} = 12x^5 - 9x^{-10}$$

$$\begin{aligned}
 * \frac{d}{dx} \left[\frac{\sqrt{x} - 2x}{\sqrt{x}} \right] &= \frac{d}{dx} [1 - 2\sqrt{x}] \\
 &= \frac{d}{dx} [1] - \frac{d}{dx} [2\sqrt{x}] = 0 - 2 \left(\frac{1}{2\sqrt{x}} \right) \\
 &= \boxed{-\frac{1}{\sqrt{x}}} .
 \end{aligned}$$

Remember that :

$$* \frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)]$$

$$* \frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c}$$

$$* \frac{d}{dx} [\sqrt{x}] = \frac{1}{2\sqrt{x}}$$

* *Although Formulas (9) and (10) are stated for sums and differences of two functions, they can be extended to any finite number of functions.*

* *Formula (8) can also be expressed in function notation as*

$$\boxed{(f + g + h)' = f' + g' + h'}$$

Example 6 :

Find dy / dx if $y = 3x^8 - 2x^5 + 6x + 1$.

Solution

$$y = 3x^8 - 2x^5 + 6x + 1$$

$$\frac{dy}{dx} = \frac{d}{dx} [3x^8 - 2x^5 + 6x + 1]$$

Remember that :

$$\begin{aligned} * \frac{d}{dx} [f(x) \pm g(x)] &= \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)] \\ &= \frac{d}{dx} [3x^8] - \frac{d}{dx} [2x^5] + \frac{d}{dx} [6x] + \frac{d}{dx} [1] \end{aligned}$$

Remember that :

$$\begin{aligned} * \frac{d}{dx} [c f(x)] &= c \frac{d}{dx} [f(x)] \\ * \frac{d}{dx} [x^r] &= r x^{r-1}, \quad \frac{d}{dx} [c] = 0 \end{aligned}$$

$$= \boxed{24x^7 - 10x^4 + 6}.$$

Higher Derivatives :

* The derivative f' of a function f is itself a function and hence may have a derivative of its own. If f' is differentiable, then its derivative is denoted by f'' and is called second derivative of f . As long as we have differentiability, we can continue the process of differentiating to obtain third, fourth, fifth, and even higher derivatives of f . These successive derivatives are denoted by

$$f', f'' = (f')', f''' = (f'')', f^{(4)} = (f''')', f^{(5)} = (f^{(4)})', \dots$$

* If $y = f(x)$, then successive derivatives can also be denoted by

$$y', y'', y''', y^{(4)}, y^{(5)}, \dots$$

* *Other common notations are*

$$\begin{aligned} y' &= \frac{dy}{dx} = \frac{d}{dx}[f(x)] \\ y'' &= \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[\frac{d}{dx}[f(x)] \right] = \frac{d^2}{dx^2}[f(x)] \\ y''' &= \frac{d^3 y}{dx^3} = \frac{d}{dx} \left[\frac{d^2}{dx^2}[f(x)] \right] = \frac{d^3}{dx^3}[f(x)] \\ &\vdots \qquad \qquad \qquad \vdots \end{aligned}$$

* *The number of times that f is differentiated is called the order of the derivative. A general n th order derivative can be denoted by*

$$\boxed{\frac{d^n y}{dx^n} = f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]} \quad (11)$$

and the value of a general n th order derivative at a specific point $x = x_0$ can be denoted by

$$\boxed{\left. \frac{d^n y}{dx^n} \right|_{x=x_0} = f^{(n)}(x_0) = \left. \frac{d^n}{dx^n}[f(x)] \right|_{x=x_0}} \quad (12)$$

Example 9 :

* *If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, then*

$$\begin{aligned} f'(x) &= 12x^3 - 6x^2 + 2x - 4 \\ f''(x) &= 36x^2 - 12x + 2 \\ f'''(x) &= 72x - 12 \\ f^{(4)}(x) &= 72 \end{aligned}$$

$$\begin{aligned}
 f^{(5)}(x) &= 0 \\
 &\vdots \\
 f^{(n)}(x) &= 0 \quad (n \geq 5)
 \end{aligned}$$

EXERCISE SET 2.3: (Home Work)

2,3 Find dy/dx .

2. $y = -3x^{12}$

3. $y = 3x^8 + 2x + 1$

10,11,13 Find $f'(x)$.

10. $f(x) = \sqrt{x} + \frac{1}{x}$

11. $f(x) = -3x^{-8} + 2\sqrt{x}$

13. $f(x) = x^e + \frac{1}{x^{\sqrt{10}}}$

18. Find $y'(1)$ if $y = \frac{x^{3/2} + 2}{x}$.

41,42 Find d^2y/dx^2 .

41. (a) $y = 7x^3 - 5x^2 + x$

(c) $y = \frac{x+1}{x}$

42. (a) $y = 4x^7 - 5x^3 + 2x$

(d) $y = (x^3 - 5)(2x + 3)$

2.4 THE PRODUCT AND QUOTIENT RULES :

Derivative of a Product :

2.4.1 Theorem (The Product Rule):

If f and g are differentiable at x , then so is the product $f \cdot g$, and

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)] \quad (1)$$

* Formula (1) can also be expressed as

$$(f \cdot g)' = f \cdot g' + g \cdot f'.$$

In words, the derivative of a product of two functions is the first function times the derivative of the second plus the second times the derivative of the first.

Example 1 :

Find dy / dx if $y = (4x^2 - 1)(7x^3 + x)$.

Solution

$$y = (4x^2 - 1)(7x^3 + x)$$

Method 1.: (Using the Product Rule)

$$\frac{dy}{dx} = \frac{d}{dx}[(4x^2 - 1)(7x^3 + x)]$$

Remember that :

$$* \frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)]$$

$$\begin{aligned}
&= (4x^2 - 1) \cdot \frac{d}{dx} [7x^3 + x] + (7x^3 + x) \cdot \frac{d}{dx} [4x^2 - 1] \\
&= (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x) \\
&= \boxed{140x^4 - 9x^2 - 1}.
\end{aligned}$$

Method 2.: (Multiplying First)

$$y = (4x^2 - 1)(7x^3 + x)$$

$$= 28x^5 + 4x^3 - 7x^3 - x = 28x^5 - 3x^3 - x$$

$$* \text{ Thus, } \frac{dy}{dx} = \frac{d}{dx} [28x^5 - 3x^3 - x] = \boxed{140x^4 - 9x^2 - 1}.$$

which agrees with the result obtained using the product rule.

Example 2 :

Find ds / dt if $s = (1+t)\sqrt{t}$.

Solution

$$s = (1+t)\sqrt{t}$$

* Applying the product rule yields

$$\frac{ds}{dt} = \frac{d}{dt} [(1+t)\sqrt{t}]$$

Remember that :

$$\begin{aligned}
&* \frac{d}{dx} [f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx} [g(x)] + g(x) \cdot \frac{d}{dx} [f(x)] \\
&= (1+t) \cdot \frac{d}{dt} [\sqrt{t}] + \sqrt{t} \cdot \frac{d}{dt} [(1+t)]
\end{aligned}$$

Remember that :

$$* \frac{d}{dx} [\sqrt{x}] = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned}
 &= (1+t) \frac{1}{2\sqrt{t}} + \sqrt{t} (1) \\
 &= \frac{1+t}{2\sqrt{t}} + \sqrt{t} \frac{2\sqrt{t}}{2\sqrt{t}} = \boxed{\frac{1+3t}{2\sqrt{t}}} .
 \end{aligned}$$

Derivative of a Quotient :

2.4.2 Theorem (The Quotient Rule):

If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and

$$\boxed{\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx} [f(x)] - f(x) \cdot \frac{d}{dx} [g(x)]}{[g(x)]^2}} . \quad (2)$$

** Formula (2) can also be expressed as*

$$\boxed{\left(\frac{f}{g} \right)' = \frac{g \cdot f' - f \cdot g'}{g^2}} .$$

In words , the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator , all divided by the denominator squared.

Example 3 :

Find $y'(x)$ for $y = \frac{x^3 + 2x^2 - 1}{x + 5}$.

Solution

$$y = \frac{x^3 + 2x^2 - 1}{x + 5}$$

* *Applying the quotient rule yields*

$$\frac{dy}{dx} = \frac{d}{dx} \left[\frac{x^3 + 2x^2 - 1}{x + 5} \right]$$

Remember that :

$$* \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx} [f(x)] - f(x) \cdot \frac{d}{dx} [g(x)]}{[g(x)]^2}$$

$$= \frac{(x + 5) \cdot \frac{d}{dx} [x^3 + 2x^2 - 1] - (x^3 + 2x^2 - 1) \cdot \frac{d}{dx} [x + 5]}{(x + 5)^2}$$

$$= \frac{(x + 5)(3x^2 + 4x) - (x^3 + 2x^2 - 1)(1)}{(x + 5)^2}$$

$$= \frac{(3x^3 + 19x^2 + 20x) - (x^3 + 2x^2 - 1)}{(x + 5)^2}$$

$$= \boxed{\frac{2x^3 + 17x^2 + 20x + 1}{(x + 5)^2}}.$$

Table 2.4.1 Rules for Differentiation:

$$\frac{d}{dx}[c] = 0$$

$$(cf)' = cf'$$

$$(f + g)' = f' + g'$$

$$(f - g)' = f' - g'$$

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

$$\frac{d}{dx}[x^r] = r x^{r-1}$$

EXERCISES SET 2.4: (Home Work)

4. Compute the derivative of the given function

$$f(x) = (x + 1)(x^2 - x + 1)$$

by (a) multiplying and then differentiating and
(b) using the product rule.

Verify that (a) and (b) yield the same result.

8,9,11,14 Find $f'(x)$.

$$8. f(x) = \left(\frac{1}{x} + \frac{1}{x^2}\right)(3x^3 + 27)$$

$$9. f(x) = (x - 2)(x^2 + 2x + 4)$$

$$11. f(x) = \frac{3x + 4}{x^2 + 1}$$

$$14. f(x) = \frac{2x^2 + 5}{3x - 4}$$

$$21. \text{ Find } dy/dx \Big|_{x=1} \text{ if } y = \frac{2x-1}{x+3}.$$

2.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS :

Using the definition of the derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

* Thus, we have obtain

$$\boxed{\frac{d}{dx} \sin x = \cos x} \quad (3)$$

* we can use the same method to derive the following formula for the derivative of $\cos x$:

$$\boxed{\frac{d}{dx} \cos x = -\sin x} \quad (4)$$

Example 1 :

Find dy/dx if $y = x \sin x$.

Solution

$$y = x \sin x$$

* Using Formula (3) and the product rule we obtain

$$\frac{dy}{dx} = \frac{d}{dx} [x \sin x]$$

Remember that :

$$\begin{aligned}
 * \quad \frac{d}{dx}[f(x) \cdot g(x)] &= f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)] \\
 &= x \cdot \frac{d}{dx}[\sin x] + \sin x \cdot \frac{d}{dx}[x]
 \end{aligned}$$

Remember that :

$$* \quad \frac{d}{dx} \sin x = \cos x$$

$$= \boxed{x \cos x + \sin x}.$$

Example 2 :

Find dy / dx if $y = \frac{\sin x}{1 + \cos x}$.

Solution

$$y = \frac{\sin x}{1 + \cos x}$$

* Using the quotient rule together with Formula (3) and (4) we obtain

$$\frac{dy}{dx} = \frac{d}{dx} \left[\frac{\sin x}{1 + \cos x} \right]$$

Remember that :

$$* \quad \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx}[f(x)] - f(x) \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$\begin{aligned}
 &= \frac{(1 + \cos x) \cdot \frac{d}{dx}[\sin x] - (\sin x) \cdot \frac{d}{dx}[1 + \cos x]}{(1 + \cos x)^2}
 \end{aligned}$$

Remember that :

$$* \quad \frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x$$

$$= \frac{(1 + \cos x)(\cos x) - (\sin x)(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1$$

$$* 1 + \tan^2 x = \sec^2 x$$

$$* \cot^2 x + 1 = \csc^2 x$$

$$= \frac{\cos x + 1}{(1 + \cos x)^2} = \boxed{\frac{1}{1 + \cos x}}.$$

* The derivatives of the remaining trigonometric functions are

$$\boxed{\frac{d}{dx} \tan x = \sec^2 x}$$

$$\boxed{\frac{d}{dx} \cot x = -\csc^2 x} \quad (5-6)$$

$$\boxed{\frac{d}{dx} \sec x = \sec x \tan x}$$

$$\boxed{\frac{d}{dx} \csc x = -\csc x \cot x} \quad (7-8)$$

* These can all be obtained using the definition of the derivative, but it is easier to use Formula (3) and (4) and apply the quotient rule to the relationships

$$\tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x}, \sec x = \frac{1}{\cos x}, \csc x = \frac{1}{\sin x}$$

* For example,

$$\frac{d}{dx} [\tan x] = \frac{d}{dx} \left[\frac{\sin x}{\cos x} \right]$$

Remember that :

$$\begin{aligned}
 * \quad \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] &= \frac{g(x) \cdot \frac{d}{dx} [f(x)] - f(x) \cdot \frac{d}{dx} [g(x)]}{[g(x)]^2} \\
 &= \frac{\cos x \cdot \frac{d}{dx} [\sin x] - \sin x \cdot \frac{d}{dx} [\cos x]}{\cos^2 x}
 \end{aligned}$$

Remember that :

$$* \quad \frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x$$

$$\begin{aligned}
 &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}
 \end{aligned}$$

Remember that :

$$* \quad \cos^2 x + \sin^2 x = 1$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Example 3 :

Find $f''(\pi/4)$ if $f(x) = \sec x$.

Solution

$$f(x) = \sec x$$

$$* \quad f'(x) = \frac{d}{dx} [\sec x] = \boxed{\sec x \tan x}.$$

Remember that :

$$* \quad \frac{d}{dx} \sec x = \sec x \tan x$$

$$* f''(x) = \frac{d}{dx} [\sec x \tan x]$$

Remember that :

$$\begin{aligned} * \frac{d}{dx} [f(x) \cdot g(x)] &= f(x) \cdot \frac{d}{dx} [g(x)] + g(x) \cdot \frac{d}{dx} [f(x)] \\ &= \sec x \cdot \frac{d}{dx} [\tan x] + \tan x \cdot \frac{d}{dx} [\sec x] \end{aligned}$$

Remember that :

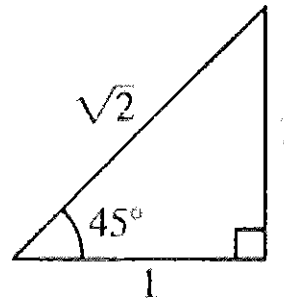
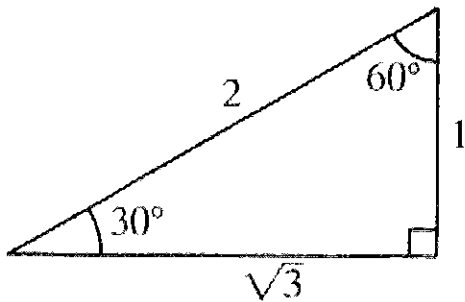
$$* \frac{d}{dx} \tan x = \sec^2 x$$

$$\begin{aligned} &= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x \\ &= \boxed{\sec^3 x + \sec x \tan^2 x} . \end{aligned}$$

$$* f''(\pi/4) = \sec^3(\pi/4) + \sec(\pi/4) \tan^2(\pi/4)$$

Remember that :

$$* \frac{\pi}{4} = \frac{180^\circ}{4} = 45^\circ, \tan \frac{\pi}{4} = 1, \sec \frac{\pi}{4} = \sqrt{2}$$



$$= (\sqrt{2})^3 + (\sqrt{2})(1)^2 = \boxed{3\sqrt{2}} .$$

EXERCISE SET 2.5: (Home Work)

1,5,8,12,14,15 Find $f'(x)$.

1. $f(x) = 4 \cos x + 2 \sin x$

5. $f(x) = \frac{5 - \cos x}{5 + \sin x}$

8. $f(x) = (x^2 + 1) \sec x$

12. $f(x) = \csc x \cot x$

14. $f(x) = \frac{\sec x}{1 + \tan x}$

15. $f(x) = \sin^2 x + \cos^2 x$

21,23 Find d^2y/dx^2 .

21. $f(x) = x \sin x - 3 \cos x$

23. $f(x) = \sin x \cos x$

26(c). Find an equation for the tangent line to the graph of $y = \sin x$ at $x = \pi/4$.

27(a). Show that $y = x \sin x$ is a solution to $y'' + y = 2 \cos x$.

2.6 THE CHAIN RULE :

Derivatives of Compositions :

2.6.1 Theorem (The Chain Rule) :

If g is differentiable and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x . Moreover, if

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}} \quad (1)$$

Example 1 :

Find dy / dx if $y = \cos(x^3)$.

Solution

$$y = \cos(x^3)$$

* Let $\boxed{u = x^3}$ and express y as $y = \cos u$. Applying Formula (1) yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}[\cos u] \cdot \frac{d}{dx}[x^3] \end{aligned}$$

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}, \quad \frac{d}{dx} \cos x = -\sin x$$

$$= (-\sin u) \cdot (3x^2)$$

$$= \left(-\sin(x^3) \right) \cdot (3x^2) = \boxed{-3x^2 \sin(x^3)}.$$

Example 2 :

Find dw / dt if $w = \tan x$ and $x = 4t^3 + t$.

Solution

$$w = \tan x \quad \text{and} \quad x = 4t^3 + t$$

* In this case the chain rule computations take the form

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\ &= \frac{d}{dx} [\tan x] \cdot \frac{d}{dt} [4t^3 + t] \end{aligned}$$

Remember that :

*

$$\begin{aligned} \frac{d}{dx} [x^r] &= r x^{r-1}, \quad \frac{d}{dx} \tan x = \sec^2 x \\ &= (\sec^2 x) \cdot (12t^2 + 1) \\ &= \left[\sec^2 (4t^3 + t) \right] \cdot (12t^2 + 1) \\ &= \boxed{(12t^2 + 1) \sec^2 (4t^3 + t)}. \end{aligned}$$

An Alternative Version of the Chain Rule :

* Formula (1) for the chain rule can be expressed in the form

$$\boxed{\frac{d}{dx} [f(g(x))] = (f \circ g)'(x) = f'(g(x)) \cdot g'(x)} \quad (2)$$

* A convenient way to remember this formula is to call f the "outside function" and g the "inside function" in the composition $f(g(x))$ and the express (2) in words as :

The derivative of $f(g(x))$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

Example 3 : (Example 1 revisited)

Find $h'(x)$ if $h(x) = \cos(x^3)$.

Solution

$$h(x) = \cos(x^3)$$

* We can think of h as a composition $f(g(x))$ in which $\boxed{g(x) = x^3}$ is the inside function and $f(x) = \cos x$ is the outside function. Thus, Formula (2) yields

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= f'(x^3) \cdot 3x^2 \\ &= -\sin(x^3) \cdot 3x^2 = \boxed{-3x^2 \sin(x^3)}. \end{aligned}$$

which agrees with the result obtained in Example 1.

Example 4 :

$$\begin{aligned} * \quad \frac{d}{dx} [\tan^2 x] &= \frac{d}{dx} [(\tan x)^2] \\ &= 2(\tan x)^1 \cdot \frac{d}{dx} (\tan x) \end{aligned}$$

Remember that :

$$* \frac{d}{dx} \tan x = \sec^2 x$$

$$= (2 \tan x) \cdot (\sec^2 x) = \boxed{2 \tan x \sec^2 x}.$$

$$* \frac{d}{dx} \left[\sqrt{x^2 + 1} \right] = \frac{1}{2\sqrt{x^2 + 1}} \cdot \frac{d}{dx} (x^2 + 1)$$

Remember that :

$$* \frac{d}{dx} [\sqrt{x}] = \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \boxed{\frac{x}{\sqrt{x^2 + 1}}}.$$

Generalized Derivative Formulas :

* *There is a useful third variation of the chain rule that strikes a middle ground between Formulas (1) and (2). If we let $u = g(x)$ in (2), then we can rewrite that formula as*

$$\boxed{\frac{d}{dx} [f(u)] = f'(u) \cdot \frac{du}{dx}} \quad (3)$$

Table 2.6.1 Generalized Derivative Formulas :

If u is a differentiable function of x , then

$$\frac{d}{dx}[u^r] = r u^{r-1} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\sin u] = \cos u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = -\sin u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \sec^2 u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\cot u] = -\csc^2 u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\sec u] = \sec u \tan u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\csc u] = -\csc u \cot u \cdot \frac{du}{dx}$$

Example 5 :

Find

(a) $\frac{d}{dx}[\sin(2x)]$

(b) $\frac{d}{dx}[\tan(x^2 + 1)]$

(c) $\frac{d}{dx}[\sqrt{x^3 + \csc x}]$

(d) $\frac{d}{dx}[x^2 - x + 2]^{3/4}$

Solution

(a) $\frac{d}{dx}[\sin(2x)]$

Taking $u = 2x$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin u] = \cos u \cdot \frac{du}{dx}$$

Remember that :

$$* \frac{d}{dx}[\sin u] = \cos u \cdot \frac{du}{dx}$$

$$= \cos 2x \cdot \frac{d}{dx}[2x] = \cos 2x \cdot 2$$

$$= \boxed{2 \cos 2x}.$$

$$(b) \frac{d}{dx}[\tan(x^2 + 1)]$$

Taking $\boxed{u = x^2 + 1}$ in the generalized derivative formula for $\tan u$ yields

$$\frac{d}{dx}[\tan(x^2 + 1)] = \frac{d}{dx}[\tan u] = \sec^2 u \cdot \frac{du}{dx}$$

Remember that :

$$* \frac{d}{dx}[\tan u] = \sec^2 u \cdot \frac{du}{dx}$$

$$= \sec^2(x^2 + 1) \cdot \frac{d}{dx}[x^2 + 1]$$

$$= \sec^2(x^2 + 1) \cdot 2x$$

$$= \boxed{2x \sec^2(x^2 + 1)}.$$

$$(c) \frac{d}{dx}[\sqrt{x^3 + \csc x}]$$

Taking $\boxed{u = x^3 + \csc x}$ in the generalized derivative formula for $\sqrt{u}$ yields

$$\frac{d}{dx}[\sqrt{x^3 + \csc x}] = \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$$

Remember that :

$$* \frac{d}{dx} [\sqrt{u}] = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{x^3 + \csc x}} \cdot \frac{d}{dx} [x^3 + \csc x]$$

Remember that :

*

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$= \frac{1}{2\sqrt{x^3 + \csc x}} \cdot (3x^2 - \csc x \cot x)$$

$$= \boxed{\frac{3x^2 - \csc x \cot x}{2\sqrt{x^3 + \csc x}}}$$

$$(d) \frac{d}{dx} [x^2 - x + 2]^{3/4}$$

Taking $\boxed{u = x^2 - x + 2}$ in the generalized derivative formula for $u^{3/4}$ yields

$$\frac{d}{dx} [x^2 - x + 2]^{3/4} = \frac{d}{dx} [u^{3/4}] = \frac{3}{4} u^{-1/4} \cdot \frac{du}{dx}$$

Remember that :

$$* \frac{d}{dx} [u^r] = r u^{r-1} \cdot \frac{du}{dx}$$

$$= \frac{3}{4} (x^2 - x + 2)^{-1/4} \cdot \frac{d}{dx} [x^2 - x + 2]$$

$$= \boxed{\frac{3}{4} (x^2 - x + 2)^{-1/4} (2x - 1)}$$

Example 6 :

Find

(a) $\frac{d}{dx} \left[\sin \left(\sqrt{1 + \cos x} \right) \right]$

(b) $\frac{d\mu}{dt}$ if $\mu = \sec \sqrt{\omega t}$ (ω constant)

Solution

(a) $\frac{d}{dx} \left[\sin \left(\sqrt{1 + \cos x} \right) \right]$

Taking $u = \sqrt{1 + \cos x}$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx} \left[\sin \left(\sqrt{1 + \cos x} \right) \right] = \frac{d}{dx} [\sin u] = \cos u \cdot \frac{du}{dx}$$

Remember that :

$$* \frac{d}{dx} [\sin u] = \cos u \cdot \frac{du}{dx}$$

$$= \cos \left(\sqrt{1 + \cos x} \right) \cdot \frac{d}{dx} \left[\sqrt{1 + \cos x} \right]$$

Remember that :

$$* \frac{d}{dx} [\sqrt{u}] = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$$

$$= \cos \left(\sqrt{1 + \cos x} \right) \cdot \frac{-\sin x}{2\sqrt{1 + \cos x}}$$

$$= \boxed{-\frac{\sin x \cos \left(\sqrt{1 + \cos x} \right)}{2\sqrt{1 + \cos x}}}$$

$$(b) \frac{d\mu}{dt} = \frac{d}{dt} [\sec \sqrt{\omega t}]$$

Taking $u = \sqrt{\omega t}$ in the generalized derivative formula for $\sec u$ yields

$$\frac{d\mu}{dt} = \frac{d}{dt} [\sec \sqrt{\omega t}] = \frac{d}{dt} [\sec u] = \sec u \tan u \cdot \frac{du}{dt}$$

Remember that :

$$* \frac{d}{dx} [\sec u] = \sec u \tan u \cdot \frac{du}{dx}$$

$$= \sec \sqrt{\omega t} \tan \sqrt{\omega t} \cdot \frac{d}{dt} [\sqrt{\omega t}]$$

Remember that :

$$* \frac{d}{dx} [\sqrt{u}] = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$$

$$= \sec \sqrt{\omega t} \tan \sqrt{\omega t} \cdot \frac{\omega}{2\sqrt{\omega t}}$$

$$= \frac{\omega \sec \sqrt{\omega t} \tan \sqrt{\omega t}}{2\sqrt{\omega t}}.$$

EXERCISE SET 2.6: (Home Work)

8,14,19,21,24 Find $f'(x)$.

$$8. f(x) = (3x^2 + 2x - 1)^6$$

$$14. f(x) = \sqrt[3]{12 + \sqrt{x}}$$

$$19. f(x) = \cos^2(3\sqrt{x})$$

$$21. f(x) = 2 \sec^2(x^7)$$

$$24. f(x) = \sqrt{3x - \sin^2(4x)}$$

27,28 Find dy/dx .

$$27. y = x^3 \sin^2(5x)$$

$$28. y = \sqrt{x} \tan^3(\sqrt{x})$$

43. Find an equation for the tangent line to the graph of $y = x \cos 3x$ at $x = \pi$.

52,53 Find d^2y/dx^2 .

$$52. y = \sin(3x^2)$$

$$53. y = \frac{1+x}{1-x}$$

CHAPTER (2)

TOPICS IN DIFFERENTIATION

3.1 IMPLICIT DIFFERENTIATION :

Functions Defined Explicitly and Implicitly :

* An equation of the form $y = f(x)$ is said to define y explicitly as a function of x because the variable y appears alone on one side of the equation and does not appear at all on the other side.

* However, sometimes functions are defined by equations in which y is not alone on one side; for example, the equation

$$yx + y + 1 = x \quad (1)$$

is not of the form $y = f(x)$, but it still defines y as a function of x since it can be rewritten as

$$y = \frac{x-1}{x+1}.$$

* Thus, we say that (1) defines y implicitly as a function of x , the function being

$$f(x) = \frac{x-1}{x+1}.$$

* An equation in x and y can implicitly define more than one function. This can occur when the graph of the equation fails the vertical line test, so it is not the graph of a function of x . For example, if we solve the equation of the circle

$$x^2 + y^2 = 1 \quad (2)$$

for y in terms of x , we obtain $y = \pm\sqrt{1-x^2}$, so we have found two functions that are defined implicitly by (2), namely,

$$f_1(x) = \sqrt{1-x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1-x^2} \quad (3)$$

* The graphs of these functions are the upper and lower semicircles of the circle $x^2 + y^2 = 1$ (Figure 3.1.1).

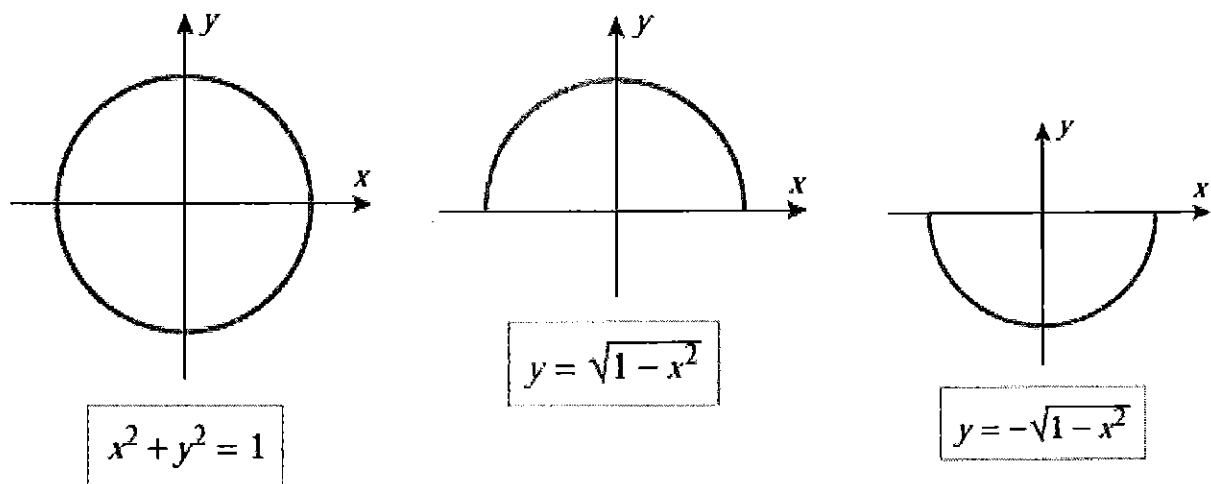


Figure 3.1.1

* This leads us to the following definition.

3.1.1 Definition :

We will say that a given equation in x and y defines the function f implicitly if the graph of $y = f(x)$ coincides with a portion of the graph of the equation.

Example 1 :

- * The graph of $x = y^2$ is not the graph of a function of x , since it does not pass the vertical line test (Figure 3.1.2).
- * However, if we solve this equation for y in terms of x , we obtain the equations $y = \sqrt{x}$ and $y = -\sqrt{x}$, whose graphs pass the vertical line test and are portions of the graph of $x = y^2$ (Figure 3.1.2).
- * Thus, the equation $x = y^2$ implicitly defines the functions
$$f_1(x) = \sqrt{x} \quad \text{and} \quad f_2(x) = -\sqrt{x}$$

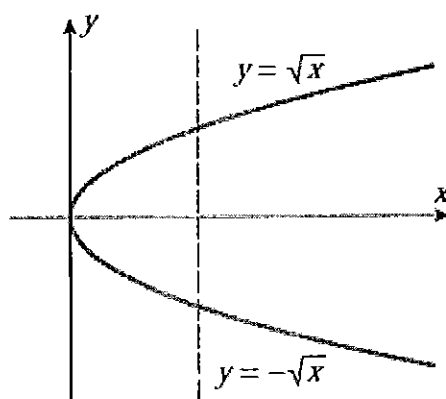


Figure 3.1.2

Implicit Differentiation :

- * In general , it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly by the equation. To illustrate this , let us consider the simple equation

$$xy = 1 \tag{5}$$

- * One way to find dy / dx is to rewrite this equation as

$$y = \frac{1}{x} \tag{6}$$

from which it follows that

$$\frac{dy}{dx} = \boxed{-\frac{1}{x^2}}. \quad (7)$$

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}$$

* Another way to obtain this derivative is to differentiate both sides of (5) before solving for y in terms of x , treating y as a differentiable function of x . With this approach we obtain

$$\frac{d}{dx}[x y] = \frac{d}{dx}[1]$$

Remember that :

$$* \frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)]$$

$$* \frac{d}{dx}[c] = 0$$

$$x \cdot \frac{d}{dx}[y] + y \cdot \frac{d}{dx}[x] = 0$$

$$x \cdot \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

If we now substitute (6) into the last expression, we obtain

$$\frac{dy}{dx} = -\frac{y}{x} = -\frac{\left(\frac{1}{x}\right)}{x} = \boxed{-\frac{1}{x^2}}.$$

which agrees with equation (7). This method of obtaining derivatives is called "implicit differentiation".

Example 2 :

Use implicit differentiation to find dy / dx if

$$5y^2 + \sin y = x^2.$$

Solution

** Differentiating both sides of*

$$5y^2 + \sin y = x^2$$

with respect to x yields

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$

Remember that :

$$* \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

$$* \frac{d}{dx}[x^r] = r x^{r-1}$$

$$5 \frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$

Remember that :

$$* \frac{d}{dx}[u^r] = r u^{r-1} \cdot \frac{du}{dx}$$

$$* \frac{d}{dx}[\sin u] = \cos u \cdot \frac{du}{dx}$$

$$5 \left(2y \frac{dy}{dx} \right) + (\cos y) \frac{dy}{dx} = 2x$$

The chain rule was used here because y is a function of x .

$$(10y + \cos y) \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}.$$

Example 3 :

Use implicit differentiation to find $d^2 y / dx^2$ if

$$4x^2 - 2y^2 = 9.$$

Solution

*** Differentiating both sides of**

$$4x^2 - 2y^2 = 9$$

with respect to x yields

$$8x - 4y \frac{dy}{dx} = 0$$

$$-4y \frac{dy}{dx} = -8x$$

$$\frac{dy}{dx} = \frac{-8x}{-4y} = \boxed{\frac{2x}{y}}. \quad (9)$$

*** Differentiating both sides of (9) yields**

Remember that :

$$* \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx} [f(x)] - f(x) \cdot \frac{d}{dx} [g(x)]}{[g(x)]^2}$$

$$\frac{d^2 y}{dx^2} = \frac{(y)(2) - (2x) \left(\frac{dy}{dx} \right)}{y^2} \quad (10)$$

*** Substituting (9) into (10) and simplifying using the original equation , we obtain**

$$\frac{d^2 y}{dx^2} = \frac{2y - 2x \left(\frac{2x}{y} \right)}{y^2}$$

$$= \frac{2y^2 - 4x^2}{y^3} = \boxed{-\frac{9}{y^3}}.$$

Example 4 :

Find the slopes of the tangent lines to the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.

Solution

$$y^2 - x + 1 = 0$$

- * *We could proceed by solving the equation for y in terms of x , and then evaluating the derivative of $y = \sqrt{x-1}$ at $(2, 1)$ and the derivative of $y = -\sqrt{x-1}$ at $(2, -1)$ (Figure 3.1.4).*

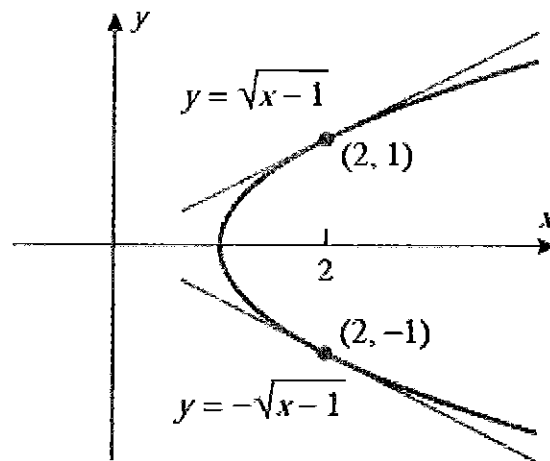


Figure 3.1.4

- * *However, implicit differentiation is more efficient since it can be used for the slopes of both lines. Differentiating implicitly yields*

$$\frac{d}{dx}[y^2 - x + 1] = \frac{d}{dx}[0]$$

Remember that :

$$* \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[y^2] - \frac{d}{dx}[x] + \frac{d}{dx}[1] = \frac{d}{dx}[0]$$

Remember that :

$$* \frac{d}{dx}[u^r] = r u^{r-1} \cdot \frac{du}{dx}$$

$$* \frac{d}{dx}[c] = 0$$

$$2y \frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \boxed{\frac{1}{2y}}.$$

* At $(2, -1)$ we have $y = -1$, and at $(2, 1)$ we have $y = 1$, so the slopes m_{tan} of the tangent lines to the curve at those points are

$$\left. \frac{dy}{dx} \right|_{\substack{x=2 \\ y=-1}} = \frac{1}{2(-1)} = \boxed{-\frac{1}{2}} \quad \text{and} \quad \left. \frac{dy}{dx} \right|_{\substack{x=2 \\ y=1}} = \frac{1}{2(1)} = \boxed{\frac{1}{2}}.$$

EXERCISE SET 3.1: (Home Work)

1. If $x + xy - 2x^3 = 2$,

(a) Find dy / dx by differentiating implicitly.

(b) Solve the equation for y as a function of x , and find dy / dx from that equation.

(c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone.

4,5,10,11 Find dy / dx by implicit differentiation.

4. $x^3 + y^3 = 3xy^2$

5. $x^2y + 3xy^3 - x = 3$

10. $\cos(xy^2) = y$

11. $\tan^3(xy^2 + y) = x$

13,17 Find d^2y / dx^2 by implicit differentiation.

13. $2x^2 - 3y^2 = 4$

17. $y + \sin y = x$

3.2 DERIVATIVES OF LOGARITHMIC FUNCTIONS :

Derivatives of Logarithmic Functions :

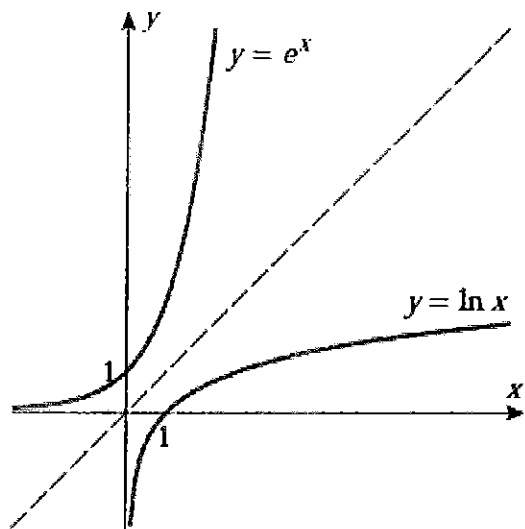
* We will establish that $f(x) = \ln x$ (natural logarithmic function) is differentiable for $x > 0$ by applying definition to $f(x)$. To evaluate the resulting limit, we will need the fact that $\ln x$ is continuous for $x > 0$, and we will need the limit

$$\boxed{\lim_{v \rightarrow 0} (1+v)^{1/v} = e} \quad (1)$$

Remember that :

* Euler's constant $e \approx 2.71828$

* $\lim_{x \rightarrow 0^+} \ln x = -\infty$, $\ln 1 = 0$, $\lim_{x \rightarrow +\infty} \ln x = +\infty$



$$\frac{d}{dx}[\ln x] = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h}$$

Remember that :

$$* \frac{d}{dx}[f(x)] = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If $a > 0$, $c > 0$, and r is any real number, then

$$* \ln(ac) = \ln a + \ln c, \ln\left(\frac{a}{c}\right) = \ln a - \ln c, \ln a^r = r \ln a$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right)$$

The quotient property of log arithm.

$$= \lim_{h \rightarrow 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right)$$

$$= \lim_{v \rightarrow 0} \frac{1}{vx} \ln(1+v)$$

Let $v = h/x$ and note that $v \rightarrow 0$ if and only if $h \rightarrow 0$.

$$= \frac{1}{x} \lim_{v \rightarrow 0} \frac{1}{v} \ln(1+v)$$

x is fixed in this limit computation, so $1/x$ can be moved through the limit sign.

$$= \frac{1}{x} \lim_{v \rightarrow 0} \ln(1+v)^{1/v}$$

The power property of logarithm.

$$= \frac{1}{x} \ln \left[\lim_{v \rightarrow 0} (1+v)^{1/v} \right]$$

$\ln x$ is continuous on $(0, +\infty)$ so we can move the limit through the function symbol.

$$= \frac{1}{x} \ln e = \frac{1}{x}$$

Since $\ln e = 1$.

*** Thus,**

$$\boxed{\frac{d}{dx} [\ln x] = \frac{1}{x}, \quad x > 0} \quad (2)$$

*** The general logarithmic function $\log_b x$, $b > 1$ is shown in Figure 0.5.6**

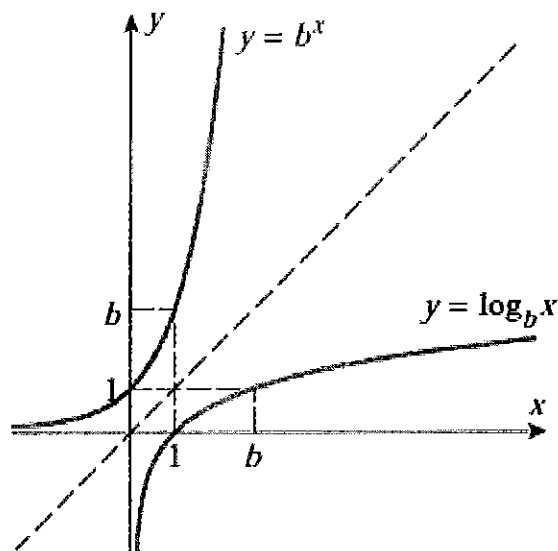


Figure 0.5.6

- * A derivative formula for the general logarithmic function $\log_b x$ can be obtained from (2) by using the Formula $\log_b x = \ln x / \ln b$

$$\frac{d}{dx}[\log_b x] = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b} \frac{d}{dx}[\ln x]$$

- * It follows from this that

$$\boxed{\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \quad x > 0} \quad (3)$$

Remember that :

*

$\log_e x = \ln x$ (natural logarithmic function),

$\log_{10} x = \log x$ (common logarithmic function)

Example 2 :

Find $\frac{d}{dx} \left[\ln(x^2 + 1) \right]$.

Solution

* Using (4) with $\boxed{u = x^2 + 1}$ we obtain

$$\frac{d}{dx} \left[\ln(x^2 + 1) \right] = \frac{1}{x^2 + 1} \cdot \frac{d}{dx} [x^2 + 1]$$

Remember that :

$$* \frac{d}{dx} [\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\begin{aligned} &= \frac{1}{x^2 + 1} \cdot 2x \\ &= \boxed{\frac{2x}{x^2 + 1}}. \end{aligned}$$

* When possible, the properties of logarithms in Theorem 0.5.2 should be used to convert products, quotients, and exponents into sums, differences, and constant multiples before differentiating a function involving logarithms.

0.5.2 Theorem (Algebraic Properties of Logarithms) :

If $b > 0$, $b \neq 1$, $a > 0$, $c > 0$, and r is any real number, then

$$(a) \log_b (ac) = \log_b a + \log_b c \quad \text{Product property}$$

$$(b) \log_b \left(\frac{a}{c} \right) = \log_b a - \log_b c \quad \text{Quotient property}$$

$$(c) \log_b (a^r) = r \log_b a$$

Power property

$$(d) \log_b \left(\frac{1}{c} \right) = -\log_b c$$

Reciprocal property

Example 3 :

Find $\frac{d}{dx} \left[\ln \left(\frac{x^2 \sin x}{\sqrt{1+x}} \right) \right]$.

Solution

* *Using properties of logarithms*

$$\frac{d}{dx} \left[\ln \left(\frac{x^2 \sin x}{\sqrt{1+x}} \right) \right] = \frac{d}{dx} \left[\ln(x^2) + \ln(\sin x) - \ln(\sqrt{1+x}) \right]$$

Remember that :

If $a > 0$, $c > 0$, and r is any real number, then

$$* \ln(ac) = \ln a + \ln c, \ln\left(\frac{a}{c}\right) = \ln a - \ln c, \ln a^r = r \ln a$$

$$= \frac{d}{dx} \left[2 \ln x + \ln(\sin x) - \frac{1}{2} \ln(1+x) \right]$$

Remember that :

$$* \frac{d}{dx} [\ln u] = \frac{1}{u} \cdot \frac{du}{dx}, \quad \frac{d}{dx} [\sin x] = \cos x$$

$$= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)}$$

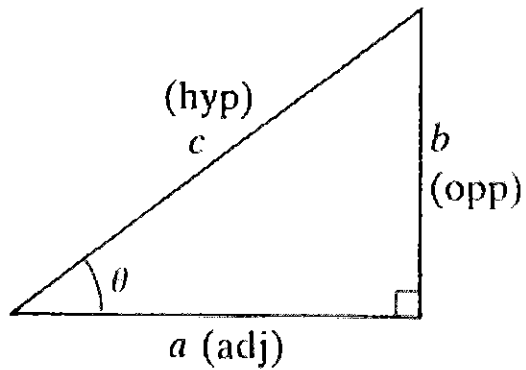
Remember that :

$$* \frac{\cos x}{\sin x} = \cot x$$

$$= \frac{2}{x} + \cot x - \frac{1}{2(1+x)}.$$

The Trigonometric Functions :

* *Of an Acute Angle θ :*



$$\sin \theta = \frac{b}{c}$$

$$\csc \theta = \frac{c}{b}$$

$$\cos \theta = \frac{a}{c}$$

$$\sec \theta = \frac{c}{a}$$

$$\tan \theta = \frac{b}{a}$$

$$\cot \theta = \frac{a}{b}$$

$$* \quad \csc \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{1}{\tan \theta},$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}, \quad \tan \theta = \frac{1}{\cot \theta}.$$

* Figure 3.2.2 shows the graph of $f(x) = \ln|x|$.

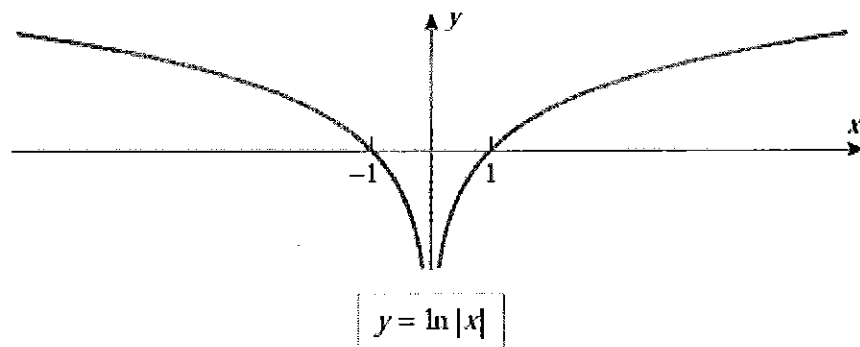


Figure 3.2.2

* This function is important because it "extends" the domain of the natural function in the sense that the values of $\ln|x|$ and $\ln x$ are the same for $x > 0$, but $\ln|x|$ is defined for all nonzero values of x , and $\ln x$ is only defined for positive values of x .

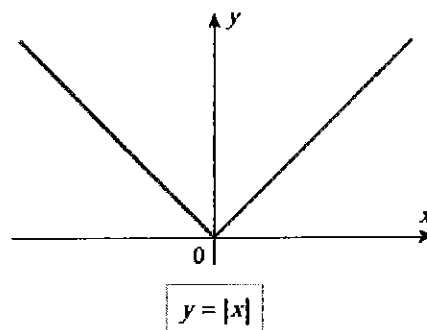
* The derivative of $\ln|x|$ for $x \neq 0$ can be obtained by considering the cases $x > 0$ and $x < 0$ separately :

Case $x > 0$. In this case $|x| = x$, so

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln x] = \frac{1}{x}$$

Remember that :

$$* |x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$



Case $x < 0$. In this case $|x| = -x$, so it follows from (4) that

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)} \cdot \frac{d}{dx}[-x] = \frac{1}{x}$$

Remember that :

$$* \frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}, \quad \frac{d}{dx}[x^r] = r x^{r-1}$$

* Since the same formula results in both cases, we have shown that

$$\boxed{\frac{d}{dx}[\ln|x|] = \frac{1}{x} \text{ if } x \neq 0} \quad (6)$$

Example 4 :

Find $\frac{d}{dx}[\ln|\sin x|]$.

Solution

* From (6) and the chain rule

$$\frac{d}{dx}[\ln|\sin x|] = \frac{1}{\sin x} \cdot \frac{d}{dx}[\sin x]$$

Remember that :

$$* \frac{d}{dx}[\ln|u|] = \frac{1}{u} \cdot \frac{du}{dx}, \quad \frac{d}{dx}[\sin x] = \cos x$$

$$= \frac{\cos x}{\sin x} = \boxed{\cot x}.$$

Remember that :

$$* \frac{\cos x}{\sin x} = \cot x$$

Logarithmic Differentiation :

* We now consider a technique called logarithmic differentiation that is useful for differentiating functions that are composed of products , quotients , and powers.

Example 5 :

Find the derivative of $y = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}$.

Solution

* The derivative of

$$y = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4} \quad (7)$$

is messy to calculate directly.

* However , if we take the natural logarithm of both sides and then use its properties , we can write

$$\ln y = \ln(x^2) + \ln(\sqrt[3]{7x-14}) - \ln[(1+x^2)^4]$$

Remember that :

If $a > 0$, $c > 0$, and r is any real number , then

$$* \ln(ac) = \ln a + \ln c , \ln\left(\frac{a}{c}\right) = \ln a - \ln c , \ln a^r = r \ln a$$

$$\ln y = 2 \ln x + \frac{1}{3} \ln(7x-14) - 4 \ln(1+x^2)$$

* Differentiating both sides with respect to x yields

$$\frac{1}{y} \frac{dy}{dx} = \frac{2}{x} + \frac{1}{3} \frac{7}{7x-14} - 4 \frac{2x}{1+x^2}$$

Remember that :

$$* \frac{d}{dx} [\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}$$

* Solving for dy / dx we obtain

$$\frac{dy}{dx} = y \left[\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2} \right]$$

* Substituting for y we obtain

$$\frac{dy}{dx} = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4} \left[\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2} \right].$$

derivatives of real powers of x :

* We know from Theorem 2.3.2 in Section 2.3 , that the differentiation formula

$$\boxed{\frac{d}{dx} [x^r] = r x^{r-1}} \quad (8)$$

holds for constant values of r . We will now use logarithmic differentiation to show that this formula hold if r is any real number (rational or irrational). In our computations we will assume that x^r is a differentiable function and the familiar laws of exponents hold for real exponents.

* Let $y = x^r$, where r is a real number. The derivative dy / dx can be obtained by logarithmic differentiation as follows:

$$\ln |y| = \ln |x^r| = r \ln |x|$$

Remember that :

* If $a > 0$ and r is any real number , then

$$\ln a^r = r \ln a$$

$$\frac{d}{dx} [\ln |y|] = \frac{d}{dx} [r \ln |x|]$$

Remember that :

$$* \frac{d}{dx} [\ln |u|] = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x} y = \frac{r}{x} x^r = r x^{r-1}$$

EXERCISE SET 3.2: (Home Work)

7,15,22,25 Find dy / dx .

$$7. y = \ln \left(\frac{x}{1+x^2} \right)$$

$$15. y = x^2 \log_2 (3-2x)$$

$$22. y = \ln(\cos x)$$

$$25. y = \log(\sin^2 x)$$

27,28 Use the properties of logarithms to perform the indicated differentiation.

$$27. \frac{d}{dx} \left[\ln \left((x-1)^3 (x^2+1)^4 \right) \right]$$

$$28. \frac{d}{dx} \left[\ln \left((\cos^2 x) \sqrt{1+x^4} \right) \right]$$

35,38 Find dy / dx using logarithmic differentiation.

$$35. y = x \sqrt[3]{1+x^2}$$

$$38. y = \frac{\sin x \cos x \tan^3 x}{\sqrt{x}}$$

41. Find the equation of the tangent line to the graph of $f(x) = \ln x$ at $x_0 = e^{-1}$.

49. Verify that $y = \ln(x+e)$ satisfies $dy / dx = e^{-y}$, with $y = 1$ when $x = 0$.

3.3 DERIVATIVES OF EXPONENTIAL AND INVERSE TRIGONOMETRIC FUNCTIONS:

Derivatives of Exponential Functions:

* Our next objective is to show that the general exponential function $y = b^x$ ($b > 0$, $b \neq 1$) (see Figure 0.5.1) is differentiable everywhere and to find its derivative.

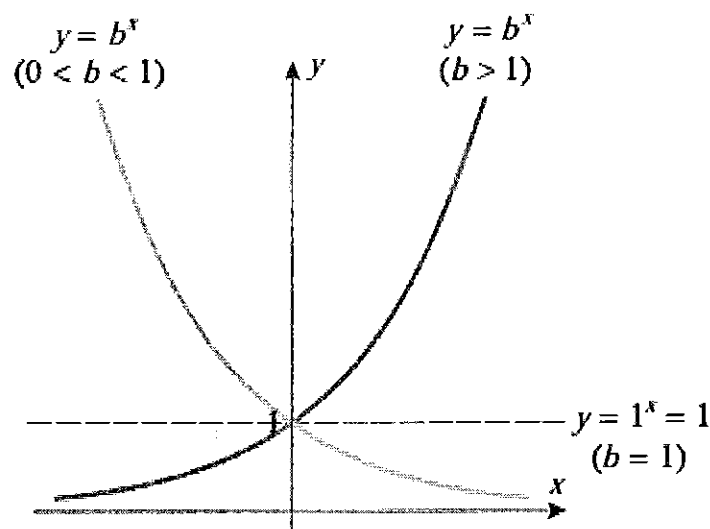


Figure 0.5.1

- * To do this, we will use the fact that b^x is the inverse of the function $f(x) = \log_b x$. We will assume that $b > 1$. With this assumption we have $\ln b > 0$, so

$$f'(x) = \frac{d}{dx} [\log_b x] = \frac{1}{x \ln b} > 0$$

for all x in the interval $(0, +\infty)$.

- * It now follows from Theorem 3.3.1 that $f^{-1}(x) = b^x$ is differentiable for all x in the range of $f(x) = \log_b x$. We know that the range of $\log_b x$ is $(-\infty, +\infty)$, so we have established that b^x is differentiable everywhere.

- * To obtain a derivative formula for b^x we rewrite $y = b^x$ as

$$x = \log_b y$$

Remember that :

$$* \log_b b^x = x, \quad f^{-1}(f(x)) = f(f^{-1}(x)) = x$$

and differentiate implicitly using Formula (5) of Section 3.2 to obtain

$$1 = \frac{1}{y \ln b} \cdot \frac{dy}{dx}$$

Solving for dy / dx and replacing y by b^x we have

$$\frac{dy}{dx} = y \ln b = b^x \ln b$$

Thus, we have shown that the derivative of general exponential function b^x is

$$\boxed{\frac{d}{dx}[b^x] = b^x \ln b} \quad (5)$$

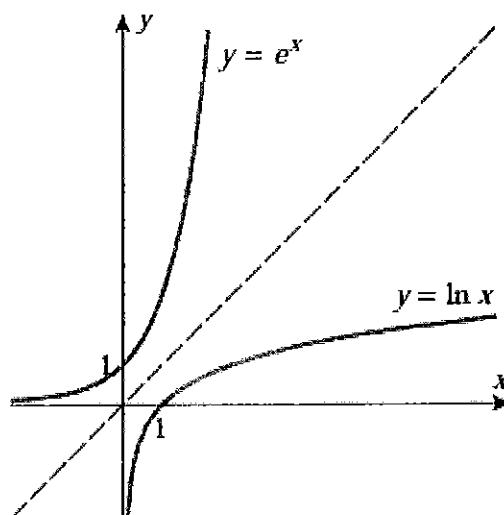
** In the special case where $b = e$ we have natural exponential function e^x , and $\ln e = 1$, so that (5) becomes*

$$\boxed{\frac{d}{dx}[e^x] = e^x} \quad (6)$$

Remember that :

$$* \ln e = 1$$

** The graph of natural exponential function e^x is shown in the following figure.*



Remember that :

* Euler's constant $e \approx 2.71828$

* Moreover, if u is a differentiable function of x , then it follows from (5) and (6) that

$$\boxed{\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx}} \quad (7)$$

and

(8)

$$\boxed{\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}} .$$

Example 3 :

The following computations use Formulas (7) and (8).

$$* \frac{d}{dx}[2^x] = \boxed{2^x \ln 2} .$$

Remember that :

$$* \frac{d}{dx}[b^x] = b^x \ln b$$

$$* \frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = \boxed{-2e^{-2x}} .$$

Remember that :

$$* \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$$

$$* \frac{d}{dx}[e^{x^3}] = e^{x^3} \cdot \frac{d}{dx}[x^3] = \boxed{3x^2 e^{x^3}} .$$

$$* \frac{d}{dx} [e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx} [\cos x] = \boxed{-(\sin x) e^{\cos x}}.$$

Remember that :

$$* \frac{d}{dx} [\cos x] = -\sin x$$

** Functions of the form $f(x) = u^v$ in which u and v are nonconstant functions of x are neither exponential functions nor power functions. Functions of this form can be differentiated using logarithmic differentiation.*

Remember that :

*If $f(x) = u^v$ in which u is a function of x and v is a constant ,
* then $f(x)$ is a power function , and*

$$f'(x) = v u^{v-1} \cdot \frac{du}{dx}.$$

For example ,

$$\frac{d}{dx} [(x^2 + 1)^5] = 5 (x^2 + 1)^4 (2x) = \boxed{10x (x^2 + 1)^4}.$$

Remember that :

*

*If $f(x) = u^v$ in which u is a constant and v is a function of x ,
then $f(x)$ is an exponential function , and*

$$f'(x) = u^v \cdot \ln u \cdot \frac{dv}{dx}.$$

For example ,

$$\frac{d}{dx} [5^{\sin x}] = 5^{\sin x} \cdot \ln 5 \cdot \frac{d}{dx} [\sin x] = \boxed{5^{\sin x} \cdot \ln 5 \cdot \cos x}$$

•

Remember that :

- If $f(x) = u^v$ in which u and v are nonconstant functions of x ,
* then $f(x)$ is neither power function nor exponential function, and use logarithmic differentiation to find $f'(x)$.

For example, $\frac{d}{dx} \left[(x^2 + 1)^{\sin x} \right]$.

Example 4 :

Use logarithmic differentiation to find $\frac{d}{dx} \left[(x^2 + 1)^{\sin x} \right]$.

Solution

* Setting $y = (x^2 + 1)^{\sin x}$.

* Taking natural logarithm of both sides

$$\ln y = \ln \left[(x^2 + 1)^{\sin x} \right]$$

Remember that :

* $\ln a^r = r \ln a$

$$\ln y = (\sin x) \ln (x^2 + 1)$$

* Differentiating both sides with respect to x yields

$$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left[(\sin x) \ln (x^2 + 1) \right]$$

Remember that :

* $\frac{d}{dx} [f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx} [g(x)] + g(x) \cdot \frac{d}{dx} [f(x)]$

$$* \frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}, \quad \frac{d}{dx}[\sin x] = \cos x$$

$$\frac{1}{y} \frac{dy}{dx} = (\sin x) \frac{1}{x^2 + 1} (2x) + (\cos x) \ln(x^2 + 1)$$

** Multiplying both sides by y*

$$\frac{dy}{dx} = y \left[\frac{2x \sin x}{x^2 + 1} + (\cos x) \ln(x^2 + 1) \right]$$

** Substituting for y*

$$\frac{dy}{dx} = \left[(x^2 + 1)^{\sin x} \left[\frac{2x \sin x}{x^2 + 1} + (\cos x) \ln(x^2 + 1) \right] \right].$$

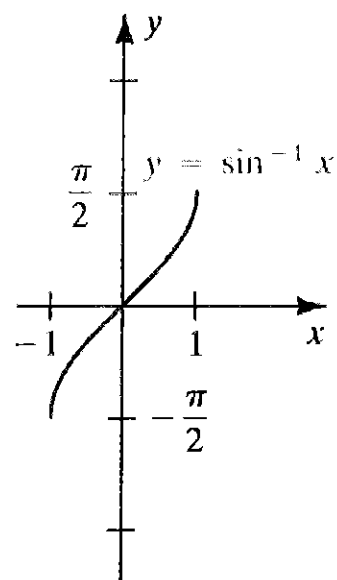
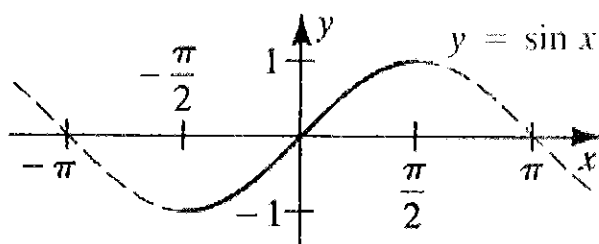
Derivatives of the Inverse Trigonometric Functions :

** Consider the function $\sin^{-1} x$. If we let $f(x) = \sin x$ ($-\pi/2 \leq x \leq \pi/2$) then it follows that $f^{-1}(x) = \sin^{-1} x$ will be differentiable at any point x where $\cos(\sin^{-1} x) \neq 0$. This is equivalent to the condition*

$$\sin^{-1} x \neq -\frac{\pi}{2} \quad \text{and} \quad \sin^{-1} x \neq \frac{\pi}{2}$$

so it follows that $\sin^{-1} x$ is differentiable on the interval $(-1, 1)$.

Remember that :



* A derivative formula for $\sin^{-1} x$ on $(-1, 1)$ can be obtained by using Formula (2) or (3) or by differentiating implicitly. Rewriting the equation $y = \sin^{-1} x$ as $x = \sin y$ and differentiating implicitly with respect to x , we obtain

$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$$

Remember that :

$$* \frac{d}{dx}[\sin u] = \cos u \cdot \frac{du}{dx}$$

$$1 = \cos y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

* This derivative formula can be simplified using identity indicated in Figure 3.3.3.

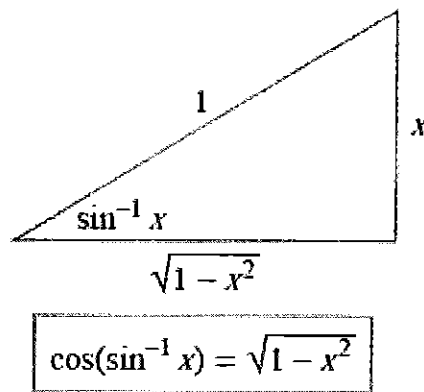


Figure 3.3.3

This yields

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

** Thus , we have shown that*

$$\frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$$

** More generally , If u is a differentiable function of x , then the chain rule produces the following generalized version of this formula :*

$$\frac{d}{dx} [\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \quad (-1 < u < 1)$$

** The method used to derive this formula can be used to obtain formulas for the remaining inverse trigonometric functions.*

$\frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx} [\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx} [\tan^{-1} x] = \frac{1}{1+x^2}$	$\frac{d}{dx} [\cot^{-1} x] = -\frac{1}{1+x^2}$

$$\frac{d}{dx} [\sec^{-1} x] = \frac{1}{|x| \sqrt{x^2 - 1}} \quad \frac{d}{dx} [\csc^{-1} x] = - \frac{1}{|x| \sqrt{x^2 - 1}}$$

* The following is a complete list of generalized formulas, each of which is valid on the natural domain of the function that multiplies du / dx .

If u is a differentiable function of x , then

$$\frac{d}{dx} [\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \quad (9)$$

$$\frac{d}{dx} [\cos^{-1} u] = - \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \quad (10)$$

$$\frac{d}{dx} [\tan^{-1} u] = \frac{1}{1+u^2} \cdot \frac{du}{dx} \quad (11)$$

$$\frac{d}{dx} [\cot^{-1} u] = - \frac{1}{1+u^2} \cdot \frac{du}{dx} \quad (12)$$

$$\frac{d}{dx} [\sec^{-1} u] = \frac{1}{|u| \sqrt{u^2 - 1}} \cdot \frac{du}{dx} \quad (13)$$

$$\frac{d}{dx} [\csc^{-1} u] = - \frac{1}{|u| \sqrt{u^2 - 1}} \cdot \frac{du}{dx} \quad (14)$$

Example 5 :

Find dy / dx if

(a) $y = \sin^{-1} (x^3)$

(b) $y = \sec^{-1} (e^x)$

Solution

$$(a) \quad y = \sin^{-1}(x^3)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}} (3x^2) = \boxed{\frac{3x^2}{\sqrt{1-x^6}}}.$$

Remember that :

$$* \quad \frac{d}{dx} [\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$(b) \quad y = \sec^{-1}(e^x)$$

$$\frac{dy}{dx} = \frac{1}{|e^x| \sqrt{(e^x)^2 - 1}} (e^x) = \boxed{\frac{1}{\sqrt{e^{2x} - 1}}}.$$

Remember that :

$$* \quad \frac{d}{dx} [\sec^{-1} u] = \frac{1}{|u| \sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

*

Since $e^x > 0$ on $(-\infty, \infty)$, then $|e^x| = e^x$, $(e^x)^2 = e^{2x}$

EXERCISES SET 3.3:

(Home Work)

21,26 Find dy / dx .

21. $y = e^{x \tan x}$

26. $y = \ln(\cos e^x)$

29. If $f(x) = \pi^{\sin x}$, find $f'(x)$ by Formula (7) and then by logarithmic differentiation.

31,32 Find dy / dx using the method of logarithmic differentiation.

31. $y = (x^3 - 2x)^{\ln x}$

32. $y = x^{\sin x}$

42. If $y = 2^{\cos x + \ln x}$, find dy / dx using any method.

43,47,56 Find dy / dx .

43. $y = \sin^{-1}(3x)$

47. $y = \tan^{-1}(x^3)$

56. $y = \csc^{-1}(e^x)$

CHAPTER (3)

THE DERIVATIVE IN GRAPHING AND APPLICATIONS

4.1 ANALYSIS OF FUNCTIONS I : INCREASE , DECREASE , AND CONCAVITY :

Increasing and Decreasing Functions :

* The terms increasing , decreasing , and constant are used to describe the behavior of a function as we travel left to right along its graph. For example , the function graphed in Figure 4.1.1 can be described as increasing to the left of $x=0$, decreasing from $x=0$ to $x=2$, increasing from $x=2$ to $x=4$, and constant to the right of $x=4$.

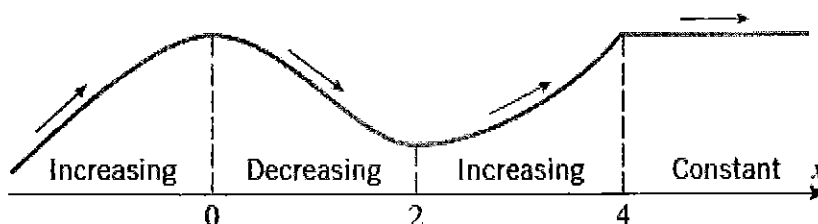


Figure 4.1.1

4.1.1 Definition :

Let f be defined on an interval , and let x_1 and x_2 denote points in that interval.

- (a) f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
- (b) f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
- (c) f is constant on the interval if $f(x_1) = f(x_2)$ for all

points x_1 and x_2 .

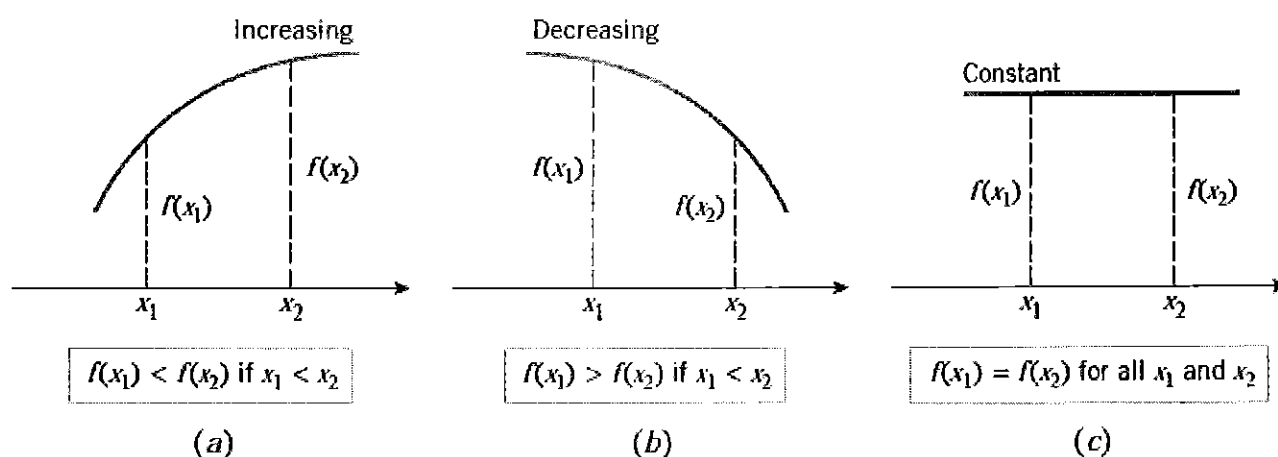


Figure 4.1.2

* Figure 4.1.3 suggests that a differentiable function f is increasing on any interval where each tangent line to its graph has positive slope, is decreasing on any interval where each tangent line to its graph has negative slope, and is constant on any interval where each tangent line to its graph has zero slope.

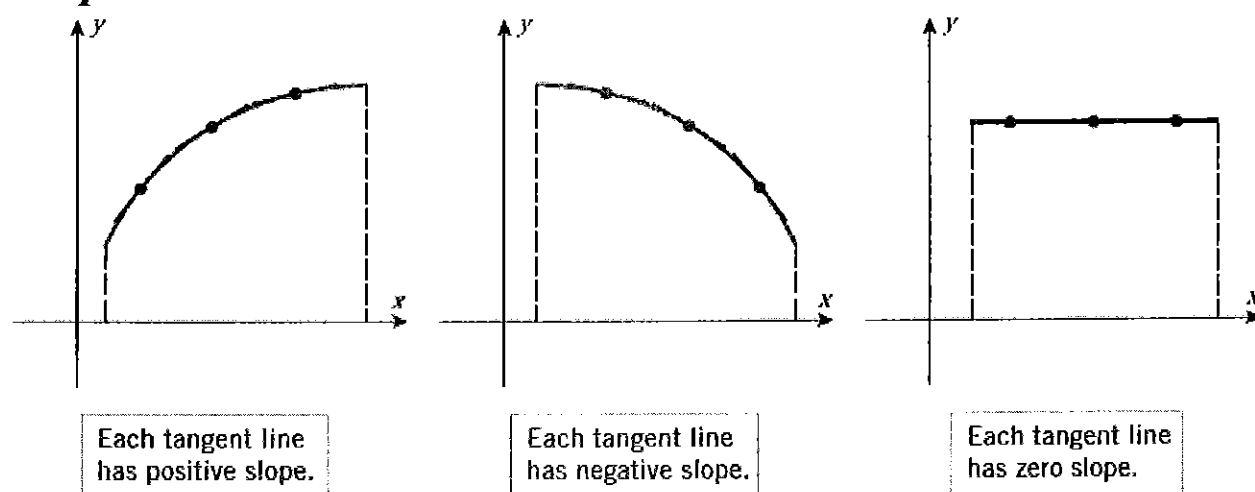


Figure 4.1.3

Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) .

- (a) If $f'(x) > 0$ for every value of x in (a, b) , then f is increasing on $[a, b]$.
- (b) If $f'(x) < 0$ for every value of x in (a, b) , then f is decreasing on $[a, b]$.
- (c) If $f'(x) = 0$ for every value of x in (a, b) , then f is constant on $[a, b]$.

* Observe that the derivative conditions in Theorem 4.1.2 are required to hold inside the interval $[a, b]$, even though the conclusions apply to the entire interval.

* Although stated for closed intervals, Theorem 4.1.2 is applicable on any interval on which f is continuous. For example, if f is continuous on $[a, +\infty)$ and $f'(x) > 0$ on $(a, +\infty)$, then f is increasing on $[a, +\infty)$; and if f is continuous on $(-\infty, +\infty)$ and $f'(x) < 0$ on $(-\infty, +\infty)$, then f is decreasing on $(-\infty, +\infty)$.

Example 1 :

Page (234)

Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.

Solution

$$f(x) = x^2 - 4x + 3$$

* The graph of f in Figure 4.1.4 suggests that f is decreasing for $x \leq 2$ and increasing for $x \geq 2$.

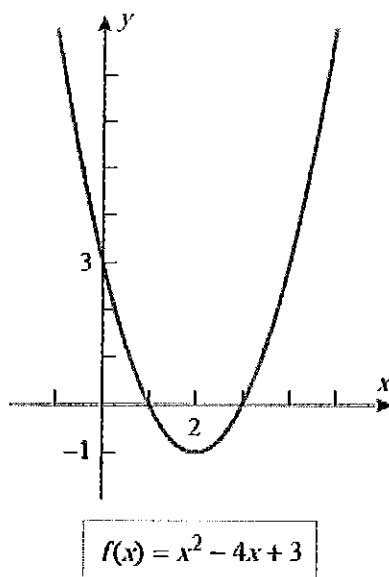


Figure 4.1.4

* To confirm this, we analyze the sign of f' . The derivative of f is

$$f'(x) = 2x - 4 = 2(x - 2)$$

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}$$

It follows that

$$f'(x) < 0 \quad \text{if} \quad x < 2$$

$$f'(x) > 0 \quad \text{if} \quad 2 < x$$

* Since f is continuous everywhere, it follows from the comment after Theorem 4.1.2 that

$f \text{ is decreasing on } (-\infty, 2]$
 $f \text{ is increasing on } [2, +\infty)$

* These conclusion are consistent with the graph of f in Figure 4.1.4.

Example 2 :

Page (234)

Find the intervals on which $f(x) = x^3$ is increasing and the intervals on which it is decreasing.

Solution

$$f(x) = x^3$$

* The graph of f in Figure 4.1.5 suggests that f is increasing over the entire x -axis.

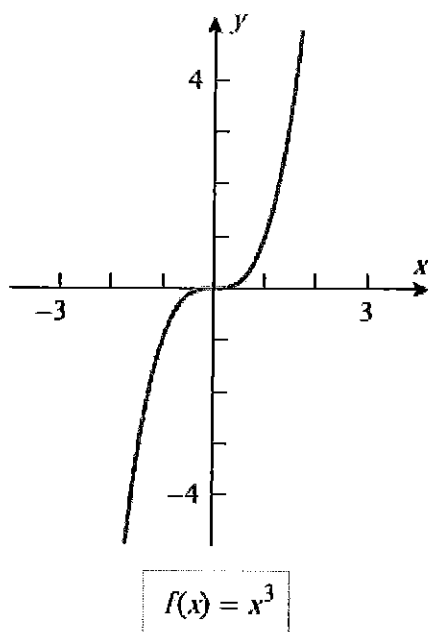


Figure 4.1.5

* To confirm this , we differentiate f to obtain $f' = 3x^2$.
Thus ,

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}$$

$$f'(x) > 0 \quad \text{if} \quad x < 0$$

$$f'(x) > 0 \quad \text{if} \quad 0 < x$$

* Since f is continuous everywhere ,

f is increasig on $(-\infty, 0]$

f is increasig on $[0, +\infty)$.

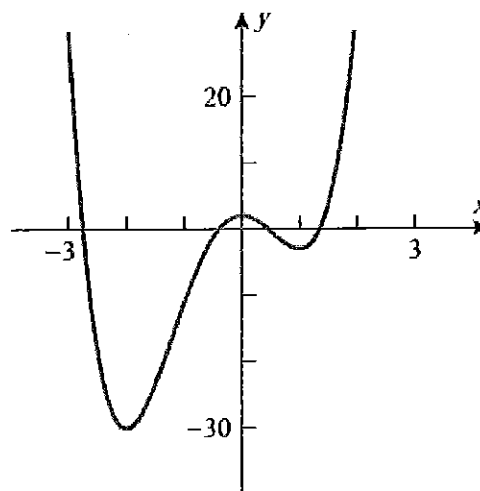
* Since f is increasing on the adjacent intervals $(-\infty, 0]$ and $[0, +\infty)$, it follows that f is increasing on their union $(-\infty, +\infty)$.

Example 3 :

Page (234)

(a) Use the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ in Figure 4.1.6 to make a conjecture about the intervals on which f is increasing or decreasing.

(b) Use Theorem 4.1.2 to determine whether your conjecture is correct.



$$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$$

Figure 4.1.6

Solution

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$$

(a) The graph suggests that the function f in Figure 4.1.6 is decreasing if $x \leq -2$, increasing if $-2 \leq x \leq 0$, decreasing if $0 \leq x \leq 1$, and increasing if $x \geq 1$.

(b) Differentiating f we obtain

$$\begin{aligned} f' &= 12x^3 + 12x^2 - 24x \\ &= 12x(x^2 + x - 2) = 12x(x+2)(x-1) \end{aligned}$$

* The sign analysis of f' in Table 4.1.1 can be obtained using the method of test points.

Table 4.1.1

Interval	$(12x)(x+2)(x-1)$	$f'(x)$	Conclusion
$x < -2$	$(-)(-)(-)$	$-$	f is decreasing on $(-\infty, -2]$
$-2 < x < 0$	$(-)(+)(-)$	$+$	f is increasing on $[-2, 0]$
$0 < x < 1$	$(+)(+)(-)$	$-$	f is decreasing on $[0, 1)$
$1 < x$	$(+)(+)(+)$	$+$	f is increasing on $[1, +\infty)$

* Figure 4.1.8 suggests two ways to characterize the concavity of a differentiable function f on an open interval :

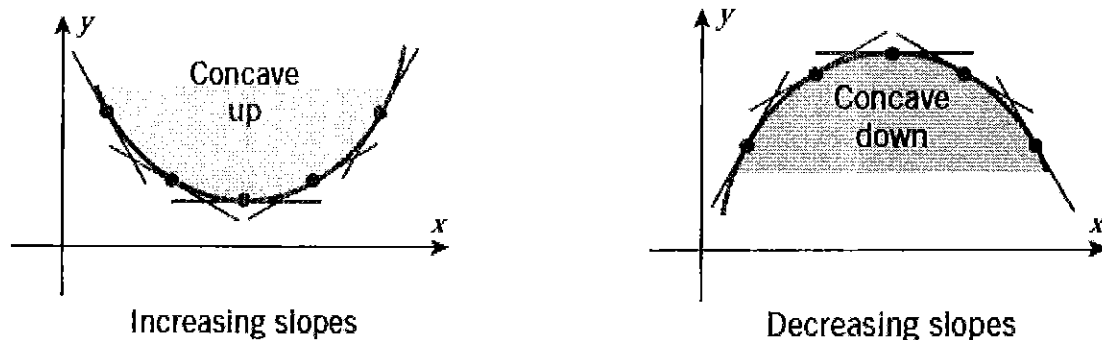


Figure 4.1.8

- f is concave up on an open interval if its tangent lines have increasing slopes on that interval and is concave down if they have decreasing slopes.
- f is concave up on an open interval if its graph lies above its tangent lines on that interval and is concave down if it lies below its tangent lines.

4.1.3 Definition :

Page (235)

If f is differentiable on an open interval, then f is said to be concave up on the open interval if f' is increasing on that interval, and f is said to be concave down on the open interval if f' is decreasing on that interval.

4.1.4 Theorem :

Page (235)

Let f be twice differentiable on an open interval.

- (a) If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.

(b) If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.

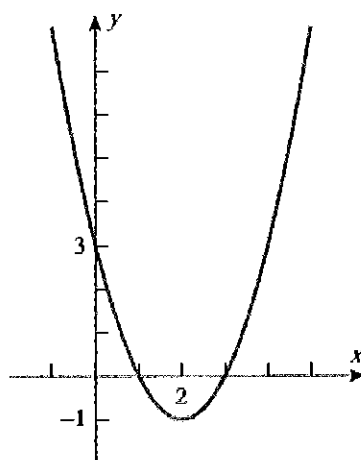
Example 4 :

Page (235)

Find the intervals on which $f(x) = x^2 - 4x + 3$ and $f(x) = x^3$ are concave up and concave down.

Solution

$$f(x) = x^2 - 4x + 3$$



$$f(x) = x^2 - 4x + 3$$

Figure 4.1.4

* Figure 4.1.4 suggests that $f(x) = x^2 - 4x + 3$ is concave up on the interval $(-\infty, +\infty)$. This is consistent with Theorem 4.1.4, since $f'(x) = 2x - 4$ and $f''(x) = 2$, so

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}$$

$f''(x) > 0$ on the interval $(-\infty, +\infty)$.

$$f(x) = x^3$$

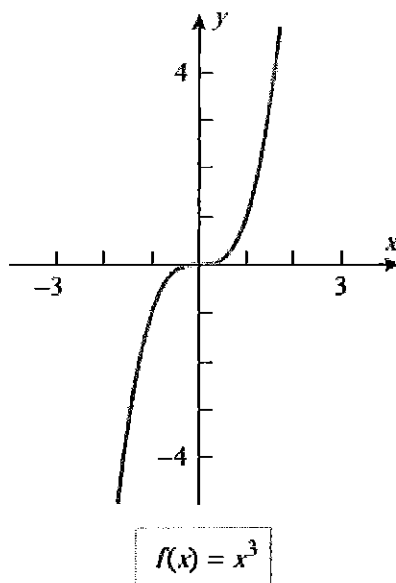


Figure 4.1.5

* Also , Figure 4.1.5 suggests that $f(x) = x^3$ is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, +\infty)$. This agrees with Theorem 4.1.4 , since $f'(x) = 3x^2$ and $f''(x) = 6x$, so $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$.

Inflection Points :

Page (236)

* We see from Example 4 and Figure 4.1.5 that the graph of $f(x) = x^3$ changes from concave down to concave up at $x = 0$. Points where a curve changes from concave up to concave down or vice versa are called inflection points.

If f is continuous on an open interval containing a value x_0 , and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an inflection point at x_0 , and we call the point $(x_0, f(x_0))$ on the graph of f an inflection point of f (Figure 4.1.9).

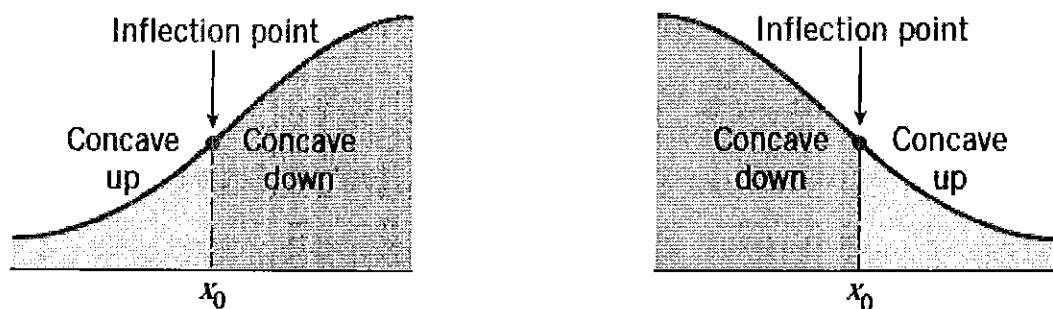
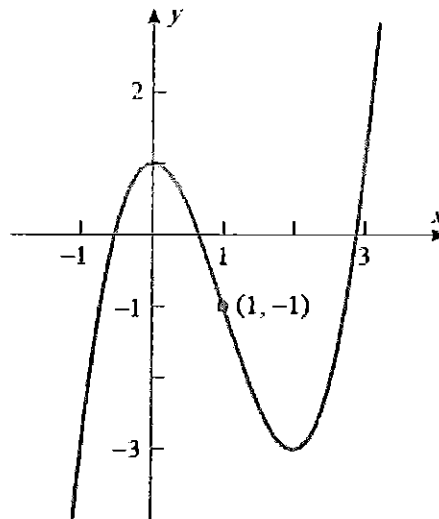


Figure 4.1.9

Example 5 :

Page (236)

Figure 4.1.10 shows the graph of the function $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.



$$f(x) = x^3 - 3x^2 + 1$$

Figure 4.1.10

Solution

$$f(x) = x^3 - 3x^2 + 1$$

* Calculate the first two derivatives of f we obtain

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}$$

$$f''(x) = 6x - 6 = 6(x - 1)$$

* The sign analysis of these derivatives is shown in the following tables :

Interval	$(3x)(x - 2)$	$f'(x)$	Conclusion
$x < 0$	$(-)(-)$	$+$	f is increasing on $(-\infty, 0]$
$0 < x < 2$	$(+)(-)$	$-$	f is decreasing on $[0, 2]$
$x > 2$	$(+)(+)$	$+$	f is increasing on $[2, +\infty)$

<i>Interval</i>	$6(x-1)$	$f''(x)$	<i>Conclusion</i>
$x < 1$	$(-)$	$-$	f is concave down on $(-\infty, 1)$
$x > 1$	$(+)$	$+$	f is concave up on $(1, +\infty)$

* The second table shows that there is an inflection point at $x = 1$, since f changes from concave down to concave up at that point. The inflection point is $(1, f(1)) = (1, -1)$. All of these conclusion are consistent with the graph of f .

EXERCISE SET 4.1:

(Home Work)

Page (241)

15,16,19,20 Find : (a) the intervals on which f is increasing ,
 (b) the intervals on which f is decreasing ,
 (c) the open intervals on which f is concave up ,
 (d) the open intervals on which f is concave down ,
 and (e) the x -coordinates of all inflection points.

15. $f(x) = x^2 - 3x + 8$

16. $f(x) = 5 - 4x - x^2$

19. $f(x) = 3x^4 - 4x^3$

20. $f(x) = x^4 - 5x^3 + 9x^2$

Relative Maxima and Minima :

* If we imagine the graph of a function f to be a two-dimensional mountain range with hills and valleys , then the tops of the hills are called "relative maxima," and the bottoms of the valleys are called "relative minima" (Figure 4.2.1).

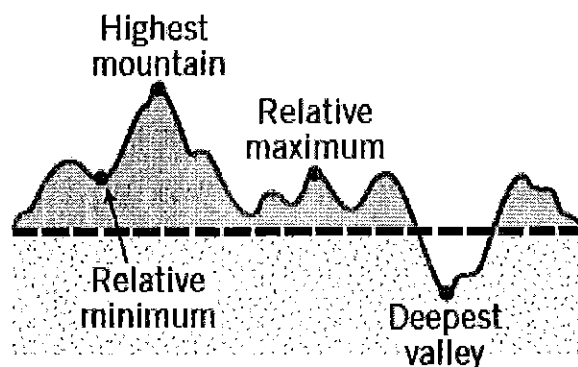


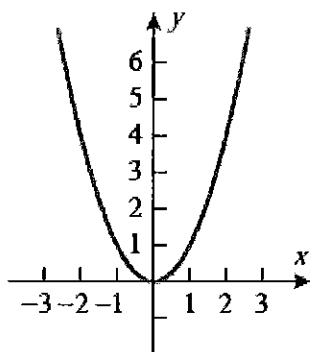
Figure 4.2.1

4.2.1 Definition :

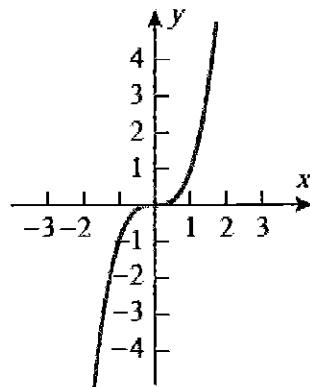
Page (244)

A function f is said to have a relative maximum at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value , that is , $f(x_0) \geq f(x)$ for all x in the interval. Similarly , f is said to have a relative minimum at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value , that is , $f(x_0) \leq f(x)$ for all x in the interval. If f has either a relative maximum or a relative minimum at x_0 , then f is said to have a relative extremum at x_0 .

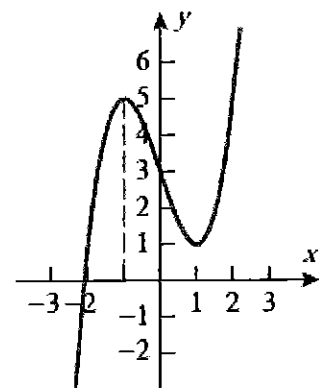
* We can see from Figure 4.2.2 that :



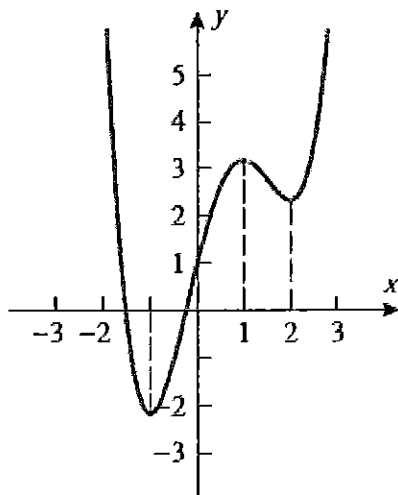
$$y = x^2$$



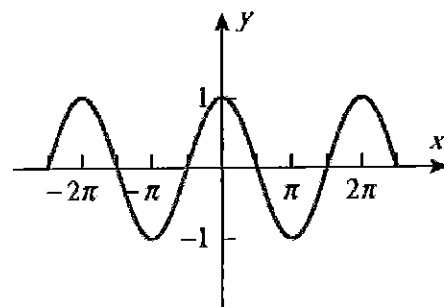
$$y = x^3$$



$$y = x^3 - 3x + 3$$



$$y = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$$



$$y = \cos x$$

Figure 4.2.2

- $f(x) = x^2$ has a relative minimum at $x = 0$ but no relative maxima.
- $f(x) = x^3$ has no relative extrema.
- $f(x) = x^3 - 3x + 3$ has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

- $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ has a relative minima at $x = -1$ and $x = 2$ and a relative maximum at $x = 1$.
- $f(x) = \cos x$ has a relative maxima at all even multiples of π and relative minima at all odd multiples of π .

* In general, we define a critical point for a function f to be a point in the domain of f at which either the graph of f has a horizontal tangent line or f is not differentiable. To distinguish between the two types of critical points we call x a stationary point of f if $f'(x) = 0$ (Figure 4.2.3).

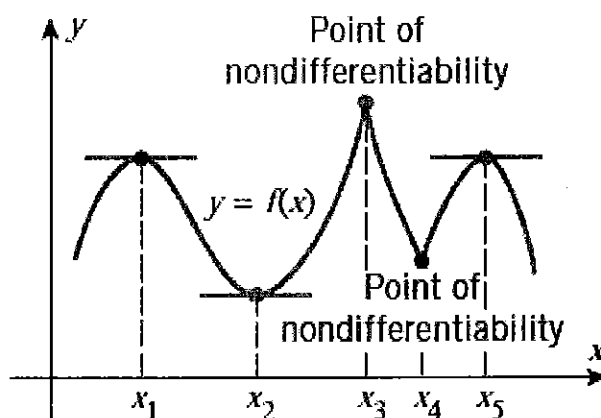


Figure 4.2.3

* The points x_1, x_2, x_3, x_4 , and x_5 are critical points. Of these, x_1, x_2 , and x_5 are stationary points.

Theorem :4.2.2

Page (245)

Suppose that f is a function defined on an open interval containing the point x_0 . If f has a relative extremum at $x = x_0$, then $x = x_0$ is a critical point of f ; that is, either $f'(x_0) = 0$ or f is not differentiable at x_0 .

Find all critical points of $f(x) = x^3 - 3x + 1$.

Solution

$$f(x) = x^3 - 3x + 1$$

* The function f , being a polynomial, is differentiable everywhere, so its critical points are all stationary points. To find these points we must solve the equation $f'(x) = 0$.

Since

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$$

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}, \quad a^2 - b^2 = (a+b)(a-b)$$

* We conclude that the critical points occur at $x = -1$ and $x = 1$. This is consistent with the graph of f in Figure 4.2.4.

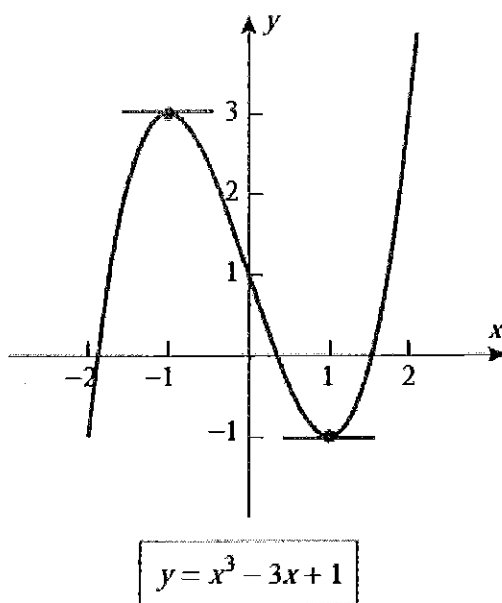


Figure 4.2.4

Example 3 :

Page (245)

Find all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$.

Solution

$$f(x) = 3x^{5/3} - 15x^{2/3}$$

* The function f is continuous everywhere and its derivative is

$$f'(x) = 5x^{2/3} - 10x^{-1/3}$$

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}$$

$$= 5x^{-1/3}(x-2)$$

$$= \boxed{\frac{5(x-2)}{x^{1/3}}}$$

* We see from this that $f'(x) = 0$ if $x = 2$ and $f'(x)$ is undefined if $x = 0$. Thus $\boxed{x = 0}$ and $\boxed{x = 2}$ are critical points and $\boxed{x = 2}$ is a stationary point. This is consistent with the graph of f shown in Figure 4.2.5.

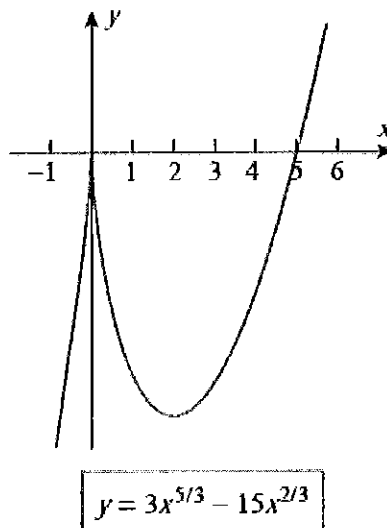


Figure 4.2.5

* *Theorem 4.2.2 asserts that the relative extrema must occur at critical points , but it does not say that a relative extremum occurs at every critical point. For example , for the eight critical points in Figure 4.2.6 , relative extrema occur at each x_0 in the top row but not at any x_0 in the bottom row. Moreover , at the critical points in the first row the derivatives have opposite signs on the two sides of x_0 , whereas at the critical points in the second row the signs of the derivatives are the same on both sides. This suggests :*

A function f has a relative extremum at those critical points where f' changes sign.

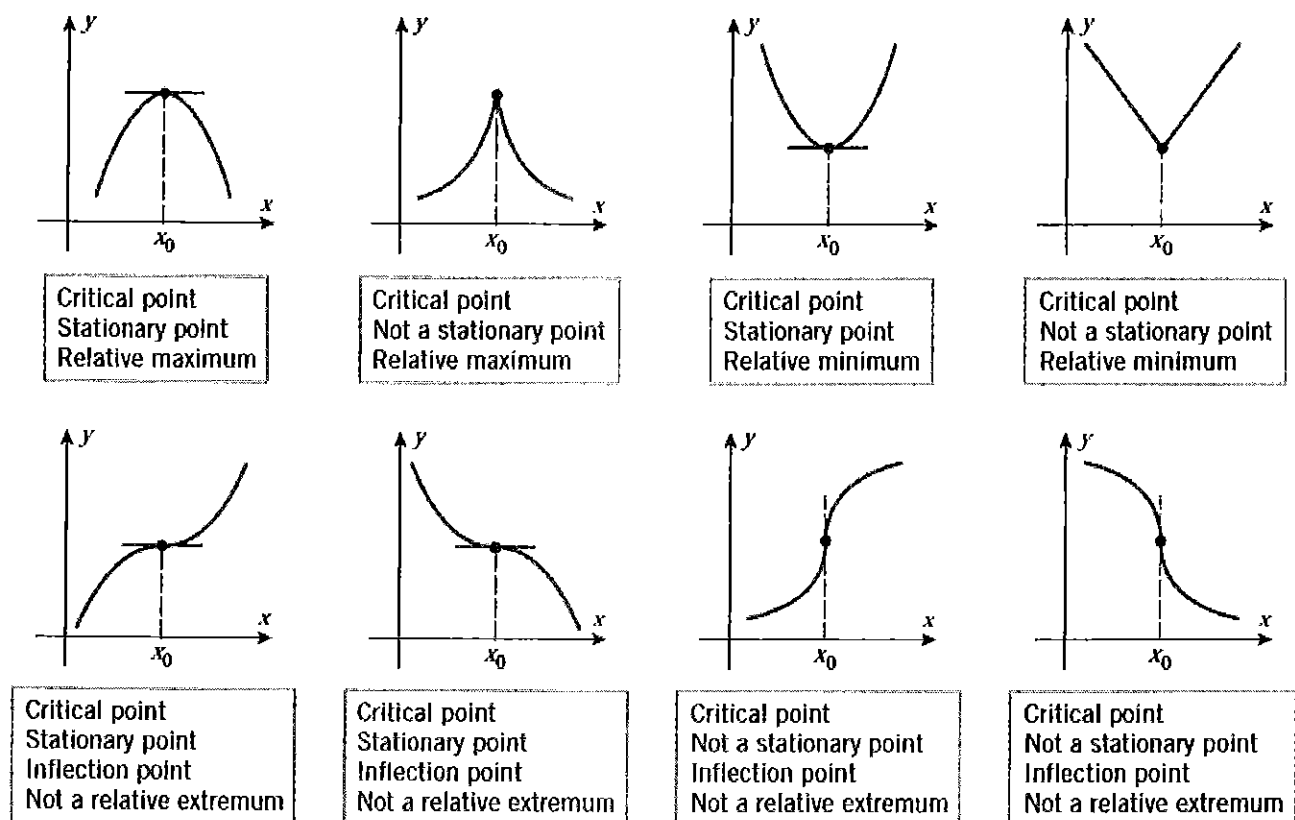


Figure 4.2.6

Suppose that f is continuous at a critical point x_0 .

- (a) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0 , then f has a relative maximum at x_0 .
- (b) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0 , then f has a relative minimum at x_0 .
- (c) If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0 , then f does not have a relative extremum at x_0 .

Example 4 :

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We showed in Example 3 that the function $f(x) = 3x^{5/3} - 15x^{2/3}$ has critical points at $x = 0$ and $x = 2$. Figure 4.2.5 suggests that f has a relative maximum at $x = 0$ and a relative minimum at $x = 2$. Confirm this using the first derivative test.

Solution

$$f(x) = 3x^{5/3} - 15x^{2/3}$$

* We showed in Example 3

$$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x - 2) = \boxed{\frac{5(x - 2)}{x^{1/3}}}$$

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}$$

* A sign analysis of this derivative is shown in Table 4.2.1

Table 4.2.1

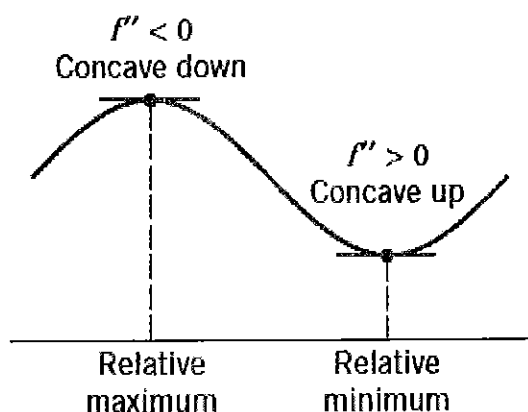
Interval	$5(x-2)/x^{1/3}$	$f'(x)$
$x < 0$	$(-)/(-)$	+
$0 < x < 2$	$(+)/(-)$	-
$x > 2$	$(+)/(+)$	+

* The sign of f' changes from + to - at $x = 0$, so there is a relative maximum at that point. The sign changes from - to + at $x = 2$, so there is a relative minimum at that point.

Second Derivative Test :

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* There is another test for relative extrema that is based on the following geometric observation : A function f has a relative maximum at a stationary point if the graph of f is concave down on an open interval containing that point, and it has a relative minimum if it is concave up (Figure 4.2.7).



4.2.4 Theorem (Second Derivative Test):

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Suppose that f is continuous at a critical point x_0 .

- (a) If $f'(x) = 0$ and $f''(x) > 0$, then f has a relative minimum at x_0 .
- (b) If $f'(x) = 0$ and $f''(x) < 0$, then f has a relative maximum at x_0 .
- (c) If $f'(x) = 0$ and $f''(x) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0 .

Example 5 :

Page (248)

Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

Solution

$$f(x) = 3x^5 - 5x^3$$

* We have

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$$

Remember that :

$$* \frac{d}{dx}[x^r] = r x^{r-1}$$

$$* a^2 - b^2 = (a+b)(a-b)$$

$$= \boxed{15x^2(x+1)(x-1)}$$

$$f''(x) = 60x^2 - 30x = 30x(2x - 1)$$

* Solving $f'(x) = 0$ yields the stationary points $x = 0$, $x = -1$, and $x = 1$.

* As shown in the following table, we can conclude from the second derivative test that f has

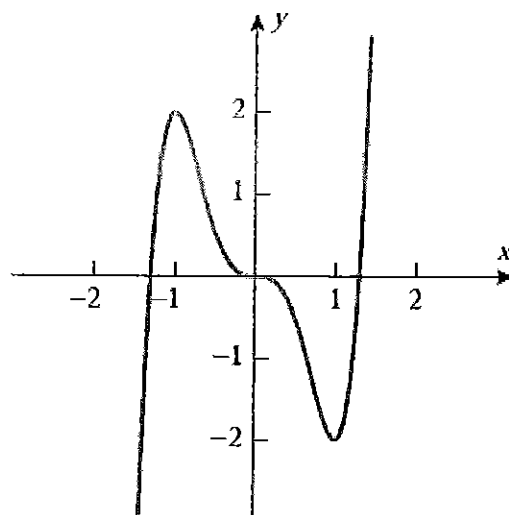
Stationary Point	$30x(2x - 1)$	$f''(x)$	Second Derivative Test
$x = -1$	-30	$-$	f has a relative maximum
$x = 0$	0	0	Inconclusive
$x = 1$	30	$+$	f has a relative minimum

a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

* The test is inconclusive at $x = 0$, so we will try the first derivative test at that point. A sign analysis of f' is given in the following table:

Interval	$15x^2(x+1)(x-1)$	$f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	$-$
$0 < x < 1$	$(+)(+)(-)$	$-$

* Since there is no sign change in f' at $x = 0$, there is neither a relative maximum nor a relative minimum at that point. All of this is consistent with the graph of f shown in Figure 4.2.8.



$$y = 3x^5 - 5x^3$$

Figure 4.2.8

EXERCISE SET 4.2:

(Home Work)

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7,8 *Locate the critical points and identify which critical points are stationary points .*

7. $f(x) = 4x^4 - 16x^2 + 17$

8. $f(x) = 3x^4 + 12x$

33,34 *Find the relative extrema using both first and second derivative tests .*

33. $f(x) = 1 + 8x - 3x^2$

34. $f(x) = x^4 - 12x^3$

37. *Use any method to find the relative extrema of the function*
 $f(x) = x^4 - 4x^3 + 4x^2$

CHAPTER (4)

INTEGRATION

5.2 THE INDEFINITE INTEGRAL :

Antiderivatives :

5.2.1 Definition :

A function F is called an antiderivative of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

* For example , the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ on the interval $(-\infty, +\infty)$ because for each x in this interval

$$F'(x) = \frac{d}{dx} \left[\frac{1}{3}x^3 \right] = x^2 = f(x)$$

* However , $F(x) = \frac{1}{3}x^3$ is not the only antiderivative of f on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f on $(-\infty, +\infty)$, since

$$G'(x) = \frac{d}{dx} \left[\frac{1}{3}x^3 + C \right] = x^2 + 0 = f(x)$$

* In general , once any single antiderivative is known , other antiderivatives can be obtained by adding constants to the known antiderivative. Thus ,

$$\frac{1}{3}x^3, \quad \frac{1}{3}x^3 + 2, \quad \frac{1}{3}x^3 - 5, \quad \frac{1}{3}x^3 + \sqrt{2}$$

are all antiderivatives of $f(x) = x^2$.

5.2.2 Theorem :

If $F(x)$ is any antiderivative of $f(x)$ on an interval, then for any constant C the function $F(x) + C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval can be expressed in the form $F(x) + C$ by choosing the constant C appropriately.

The Indefinite Integral :

* The process of finding antiderivatives is called antidifferentiation or integration. Thus, if

$$\boxed{\frac{d}{dx}[F(x)] = f(x)} \quad (1)$$

then integrating (or antidifferentiating) the function $f(x)$ produces an antiderivative of the form $F(x) + C$. To emphasize this process, Equation (1) is recast using integral notation,

$$\boxed{\int f(x) dx = F(x) + C} \quad (2)$$

where C is understood to represent an arbitrary constant. It is important to note that (1) and (2) are just different notation to express the same fact. For example

$$\int x^2 dx = \frac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2$$

* *Note that if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$ back again. Thus,*

$$\boxed{\frac{d}{dx} \left[\int f(x) dx \right] = f(x)} \quad (3)$$

* *The expression $\int f(x) dx$ is called an indefinite integral. The function $f(x)$ is called the integrand, and the constant C is called the constant of integration. Equation (2) should be read as:*

The integral of $f(x)$ with respect to x is equal to $F(x)$ plus a constant C .

* *The differential symbol, dx , in the differentiation and antidifferentiation operations*

$$\frac{d}{dx} [\quad] \quad \text{and} \quad \int [\quad] dx$$

serves to identify the independent variable.

* *Here are some examples of derivative formulas and their equivalent integration formulas :*

<i>Derivative Formula</i>	<i>Equivalent Integration Formula</i>
$\frac{d}{dx} [x^3] = 3x^2$	$\int 3x^2 dx = x^3 + C$
$\frac{d}{dx} [\sqrt{x}] = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}} dx = \sqrt{x} + C$
$\frac{d}{dt} [\tan t] = \sec^2 t$	$\int \sec^2 t dt = \tan t + C$

$\frac{d}{du} [u^{3/2}] = \frac{3}{2} u^{1/2}$	$\int \frac{3}{2} u^{1/2} du = u^{3/2} + C$
--	---

* For simplicity, the dx is sometimes absorbed into the integrand. For example,

$$\begin{aligned} \int 1 \, dx & \quad \text{can be written as} \quad \int dx \\ \int \frac{1}{x^2} \, dx & \quad \text{can be written as} \quad \int \frac{dx}{x^2} \end{aligned}$$

Integration Formulas :

* Some of the most important basic integration formulas are given in Table 5.2.1.

Table 5.2.1
Integration Formulas

Derivative Formula	Equivalent Integration Formula
1. $\frac{d}{dx} [x] = 1$	$\int dx = x + C$
2. $\frac{d}{dx} \left[\frac{x^{r+1}}{r+1} \right] = x^r \quad (r \neq -1)$	$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\frac{d}{dx} [\sin x] = \cos x$	$\int \cos x \, dx = \sin x + C$
4. $\frac{d}{dx} [-\cos x] = \sin x$	$\int \sin x \, dx = -\cos x + C$
5. $\frac{d}{dx} [\tan x] = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
6. $\frac{d}{dx} [-\cot x] = \csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$

7. $\frac{d}{dx} [\sec x] = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
8. $\frac{d}{dx} [-\csc x] = \csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
9. $\frac{d}{dx} [e^x] = e^x$	$\int e^x \, dx = e^x + C$
10. $\frac{d}{dx} \left[\frac{b^x}{\ln b} \right] = b^x \, (0 < b, b \neq 1)$	$\int b^x \, dx = \frac{b^x}{\ln b} + C \, (0 < b, b \neq 1)$
11. $\frac{d}{dx} [\ln x] = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln x + C$
12. $\frac{d}{dx} [\tan^{-1} x] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$
13. $\frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$
14. $\frac{d}{dx} [\sec^{-1} x] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$

** Note that the second integration formula in Table 5.2.1 will be easier to remember if you express it in words :*

To integrate a power of x (other than -1), add 1 to the exponent and divide by the new exponent.

Example 1 :

Evaluate the following integrals

$$* \int x^2 \, dx = \boxed{\frac{x^3}{3} + C} \quad \boxed{r=2}$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$* \int x^3 dx = \boxed{\frac{x^4}{4} + C} \quad \boxed{r=3}$$

$$* \int \frac{1}{x^5} dx = \int x^{-5} dx \quad \boxed{r=-5}$$
$$= \frac{x^{-5+1}}{-5+1} + C = \boxed{-\frac{1}{4x^4} + C}$$

$$* \int \sqrt{x} dx = \int x^{\frac{1}{2}} dx \quad \boxed{r=\frac{1}{2}}$$
$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3} x^{\frac{3}{2}} + C = \boxed{\frac{2}{3} (\sqrt{x})^3 + C}$$

Properties of the Indefinite Integral :

5.2.3 Theorem :

Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that c is a constant. Then :

(a) A constant factor can be moved through an integral sign ; that is ,

$$\boxed{\int c f(x) dx = c F(x) + C}$$

(b) An antiderivative of a sum is the sum of the antiderivatives ; that is ,

$$\boxed{\int [f(x) + g(x)] dx = F(x) + G(x) + C}$$

(c) *An antiderivative of a difference is the difference of the antiderivatives ; that is ,*

$$\int [f(x) - g(x)] dx = F(x) - G(x) + C$$

Example 2 :

Evaluate

(a) $\int 4 \cos x \, dx$

(b) $\int (x + x^2) \, dx$

Solution

(a) $\int 4 \cos x \, dx$, since $F(x) = \sin x$ is an antiderivative for $f(x) = \cos x$ (Table 5.2.1) , we obtain

$$\int 4 \cos x \, dx = 4 \int \cos x \, dx = \boxed{4 \sin x + C} .$$

Remember that :

$$* \int c f(x) \, dx = c \int f(x) \, dx = c F(x) + C$$

$$* \int \cos x \, dx = \sin x + C$$

(b) $\int (x + x^2) \, dx$, from Table 5.2.1 we obtain

$$\int (x + x^2) \, dx = \int x \, dx + \int x^2 \, dx = \boxed{\frac{x^2}{2} + \frac{x^3}{3} + C} .$$

Remember that :

$$* \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx \\ = F(x) + G(x) + C$$

$$* \int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

* Parts (b) and (c) of Theorem 5.2.3 can be extended to more than two functions, which in combination with part (a) results in the following general formula :

$$\boxed{\int [c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)] dx} \\ = c_1 \int f_1(x) dx + c_2 \int f_2(x) dx + \dots + c_n \int f_n(x) dx \quad (7)$$

Example 3 :

Evaluate $\int (3x^6 - 2x^2 + 7x + 1) dx$.

Solution

$$\begin{aligned} \int (3x^6 - 2x^2 + 7x + 1) dx \\ = 3 \int x^6 dx - 2 \int x^2 dx + 7 \int x dx + \int 1 dx \\ = \boxed{\frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C} \end{aligned}$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

* Sometimes it is useful to rewrite an integrand in a different form (simple form) before performing the integration. This is illustrated in the following example.

Example 4 :

Evaluate

$$(a) \int \frac{\cos x}{\sin^2 x} dx$$

$$(b) \int \frac{t^2 - 2t^4}{t^4} dt$$

$$(c) \int \frac{x^2}{x^2 + 1} dx$$

Solution

$$(a) \int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{\sin x} \frac{\cos x}{\sin x} dx$$

Remember that :

$$* \frac{1}{\sin x} = \csc x, \quad \frac{\cos x}{\sin x} = \cot x$$

$$= \int \csc x \cot x dx = \boxed{-\csc x + C}.$$

Remember that :

$$* \int \csc x \cot x dx = -\csc x + C$$

$$(b) \int \frac{t^2 - 2t^4}{t^4} dt = \int \left(\frac{1}{t^2} - 2 \right) dt$$

Remember that :

$$* \frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c}, \quad \frac{x^n}{x^m} = x^{n-m}$$

$$= \int (t^{-2} - 2) dx = \frac{t^{-1}}{-1} - 2t + C$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \boxed{-\frac{1}{t} - 2t + C}.$$

(c) $\int \frac{x^2}{x^2 + 1} dx$, by adding and subtracting 1 from the numerator of the integrand , we can rewrite the integrand in a form in which Formula 1 and 12 of Table 5.2.1 can be applied :

$$\int \frac{x^2}{x^2 + 1} dx = \int \frac{(x^2 + 1) - 1}{x^2 + 1} dx$$

Remember that :

$$* \frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c}$$

$$= \int \left(1 - \frac{1}{x^2 + 1} \right) dx = \boxed{x - \tan^{-1} x + C}.$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$* \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

Integral Curves :

- * *Graphs of antiderivatives of a function f are called integral curves of f . We know from Theorem 5.2.2 that if $y = F(x)$ is any integral curve of $f(x)$, then all other curves are vertical translations of this curve, since they have equations of the form $y = F(x) + C$.*
- * *For example, $y = \frac{1}{3}x^3$ is one integral curve for $f(x) = x^2$, so all other integral curves have equations of the form $y = \frac{1}{3}x^3 + C$; conversely, the graph of any equation of this form is an integral curve (Figure 5.2.1).*

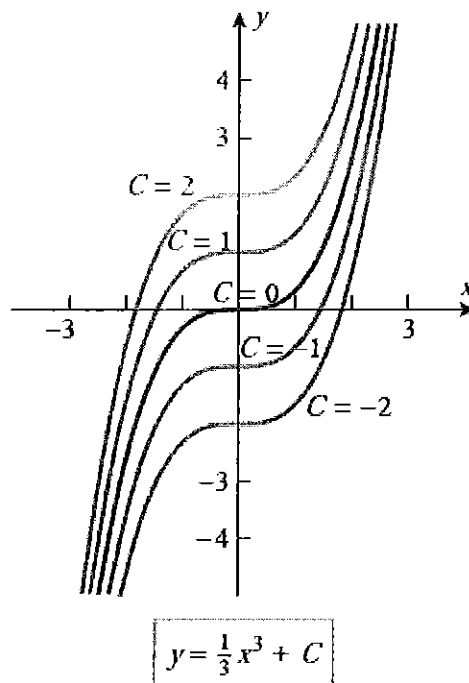


Figure 5.2.1

Example 5 :

Suppose that a curve $y = f(x)$ in the xy -plane has the property that at each point (x, y) on the curve, the tangent line has slope x^2 . Find an equation for the curve given that it passes through the point $(2, 1)$.

Solution

** Since the slope of the line tangent to $y = f(x)$ is dy/dx , we have $dy/dx = x^2$, and*

$$y = \int x^2 dx = \frac{1}{3}x^3 + C$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

* Since the curve passes through $(2,1)$, a specific value for C can be found by using the fact that $y=1$ if $x=2$. Substituting these values in the above equation yields

$$1 = \frac{1}{3}(2)^3 + C \quad \text{or} \quad C = -\frac{5}{3}$$

* So an equation of the curve is

$$y = \frac{1}{3}x^3 - \frac{5}{3}$$

(Figure 5.2.2).

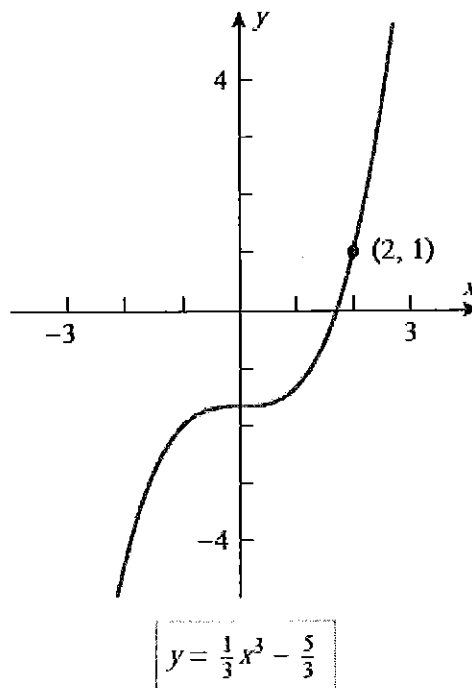


Figure 5.2.2

EXERCISE SET 5.2:*(Home Work)*

15,19,21,23, Evaluate the integral and check your answer by
25,29,32,33 differentiating.

15. $\int x(1+x^3) dx$

19. $\int \frac{x^5 + 2x^2 - 1}{x^4} dx$

21. $\int \left[\frac{2}{x} + 3e^x \right] dx$

23. $\int [3 \sin x - 2 \sec^2 x] dx$

25. $\int \sec x (\sec x + \tan x) dx$

29. $\int \frac{\sin x}{\cos^2 x} dx$

32. $\int \frac{\sec x + \cos x}{2 \cos x} dx$

33. $\int \left[\frac{1}{2\sqrt{1-x^2}} - \frac{3}{1+x^2} \right] dx$

5.3 INTEGRATION BY SUBSTITUTION :**u-Substitution :**

* The method of substitution can be motivated by examining the chain rule from the view point of antidifferentiation. For this purpose, suppose that F is an antiderivative of f and that g is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$\frac{d}{dx} [F(g(x))] = F'(g(x)) g'(x)$$

which we can write in integral form as

$$\int F'(g(x)) g'(x) dx = F(g(x)) + C \quad (1)$$

or since F is an antiderivative of f ,

$$\int f(g(x)) g'(x) dx = F(g(x)) + C \quad (2)$$

* For our purpose it will be useful to let $u = g(x)$ and to write $du/dx = g'(x)$ in the differential form $du = g'(x) dx$. With this notation (2) can be expressed as

$$\int f(u) du = F(u) + C \quad (3)$$

The process of evaluating an integral of form (2) by converting it into form (3) with the substitution

$$u = g(x) \text{ and } du = g'(x) dx$$

is called the method of u -substitution.

Example 1:

Evaluate $\int (x^2 + 1)^{50} \cdot 2x dx$.

Solution

$$\int (x^2 + 1)^{50} \cdot 2x dx$$

* If we let

$$u = x^2 + 1, \text{ then}$$

$du / dx = 2x$,
 which implies that $du = 2x \, dx$.

Thus , the given integral can be written as

$$\int (x^2 + 1)^{50} \cdot 2x \, dx = \int u^{50} \, du$$

Remember that :

$$* \int u^r \, dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{u^{51}}{51} + C = \boxed{\frac{(x^2 + 1)^{51}}{51} + C} .$$

Guidelines for u-Substitution :

Step 1. Look for some composition $f(g(x))$ within the integrand for which the substitution

$$\boxed{u = g(x)} \quad , \quad du = g'(x) \, dx$$

produces an integral that is expressed entirely in terms of u and its differential du . This may or may not be possible.

Step 2. If you are successful in Step 1 , then try to evaluate the resulting integral in terms of u . Again , this may or may not be possible.

Step 3. If we are successful in Step 2 , then replace u by $g(x)$ to express your final answer in terms of x .

Easy to Recognize Substitutions :

* *The easiest substitutions occur when the integrand is the derivative of a known function, except for a constant added to or subtracted from the independent variable.*

Example 2 :

Evaluate

$$(a) \int \sin(x+9) dx.$$

$$(b) \int (x-8)^{23} dx.$$

Solution

$$(a) \int \sin(x+9) dx.$$

* If we let $\boxed{u = x + 9}$, then
 $du = dx$.

Thus, the given integral can be written as

$$\int \sin(x+9) dx = \int \sin u du$$

Remember that :

$$* \int \sin u du = -\cos u + C$$

$$= -\cos u + C = \boxed{-\cos(x+9) + C}.$$

$$(b) \int (x-8)^{23} dx.$$

* If we let $\boxed{u = x - 8}$, then
 $du = dx$.

Thus, the given integral can be written as

$$\int (x-8)^{23} dx = \int u^{23} du$$

Remember that :

$$* \int u^r dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{u^{24}}{24} + C = \boxed{\frac{(x-8)^{24}}{24} + C}.$$

* Another easy u -substitution occurs when the integrand is the derivative of a known function, except for a constant that multiplies or divides the independent variable. The following example illustrates two ways to evaluate such integrals.

Example 3 :

Evaluate $\int \cos 5x \, dx$.

Solution

$$\int \cos 5x \, dx$$

* If we let

$$\boxed{u = 5x}, \text{ then}$$

$$du = 5 \, dx,$$

which implies that $\frac{1}{5} du = dx$.

Thus, the given integral can be written as

$$\int \cos 5x \, dx = \frac{1}{5} \int \cos u \, du$$

Remember that :

$$* \int \cos u \, du = \sin u + C$$

$$= \frac{1}{5} \sin u + C = \boxed{\frac{1}{5} \sin 5x + C}.$$

* More generally, if the integrand is a composition of the form $f(ax+b)$, where $f(x)$ is an easy to integrate function, then the substitution $\boxed{u = ax+b}$, $du = a \, dx$ will work.

Example 4 :

Evaluate $\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5}.$

Solution

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5}$$

* *If we let*

$$\boxed{u = \frac{1}{3}x - 8}, \text{ then}$$

$$du = \frac{1}{3}dx,$$

which implies that $3 du = dx.$

Thus, the given integral can be written as

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5} = 3 \int \frac{1}{u^5} du = 3 \int u^{-5} du$$

Remember that :

$$* \int u^r dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= 3 \frac{u^{-4}}{-4} + C = \boxed{-\frac{3}{4} \left(\frac{1}{3}x - 8\right)^{-4} + C}.$$

Example 5 :

Evaluate $\int \frac{dx}{1+3x^2}.$

Solution

$$\int \frac{dx}{1+3x^2} = \int \frac{dx}{1+(\sqrt{3}x)^2}$$

* *Substituting* $\boxed{u = \sqrt{3}x}$, then

$$du = \sqrt{3} dx, \text{ so } \frac{1}{\sqrt{3}} du = dx$$

Thus,

$$\int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}} \int \frac{1}{1+u^2} du$$

Remember that :

$$* \int \frac{1}{1+u^2} du = \tan^{-1} u + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} u + C = \boxed{\frac{1}{\sqrt{3}} \tan^{-1} (\sqrt{3}x) + C}.$$

* *With the help of Theorem 5.2.3, a complicated integral can sometimes be computed by expressing it as a sum of simpler integrals.*

Example 6 :

Evaluate $\int \left(\frac{1}{x} + \sec^2 \pi x \right) dx.$

Solution

$$\begin{aligned} \int \left(\frac{1}{x} + \sec^2 \pi x \right) dx &= \int \frac{1}{x} dx + \int \sec^2 \pi x dx \\ &= \ln |x| + \int \sec^2 \pi x dx \end{aligned}$$

Remember that :

$$* \int \frac{1}{x} dx = \ln |x| + C$$

* To evaluate $\int \sec^2 \pi x \, dx$, substituting

$$\boxed{u = \pi x} \text{ , then}$$

$$du = \pi dx \text{ , so } \frac{1}{\pi} du = dx$$

Thus ,

$$\int \left(\frac{1}{x} + \sec^2 \pi x \right) dx = \ln |x| + \frac{1}{\pi} \int \sec^2 u \, du$$

Remember that :

$$* \int \sec^2 u \, du = \tan u + C$$

$$= \ln |x| + \frac{1}{\pi} \tan u + C$$

$$= \boxed{\ln |x| + \frac{1}{\pi} \tan \pi x + C} .$$

* The next four examples illustrate a substitution where $g(x)$ is a nonlinear function.

$$\boxed{u = g(x)}$$

Example 7 :

Evaluate $\int \sin^2 x \cos x \, dx$.

Solution

$$\int \sin^2 x \cos x \, dx = \int (\sin x)^2 \cos x \, dx$$

* Substituting $\boxed{u = \sin x}$, then

$$du = \cos x \, dx .$$

Thus ,

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du$$

Remember that :

$$* \int u^r dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{u^3}{3} + C = \boxed{\frac{\sin^3 x}{3} + C}.$$

Example 8 :

Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$.

Solution

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int e^{\sqrt{x}} \frac{1}{\sqrt{x}} dx$$

* If we let $\boxed{u = \sqrt{x}}$, then

$$du = \frac{1}{2\sqrt{x}} dx, \text{ so } 2 du = \frac{1}{\sqrt{x}} dx.$$

Thus,

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^u du$$

Remember that :

$$* \int e^u du = e^u + C$$

$$= 2 e^u + C = \boxed{2 e^{\sqrt{x}} + C}.$$

Less Apparent Substitutions :

* The method of substitution is relatively straightforward, provided the integrand contains an easily recognized composition $f(g(x))$ and the remainder of the integrand is a constant multiple of $g'(x)$. If this is not the case, the method may still apply but may require more computation.

Example 11 :

Evaluate $\int x^2 \sqrt{x-1} dx$.

Solution
$$\int x^2 \sqrt{x-1} dx$$

* If we let $\boxed{u = x-1}$, then $u+1 = x$ and $du = dx$

Thus ,

$$\int x^2 \sqrt{x-1} dx = \int (u+1)^2 \sqrt{u} du$$

Remember that :

$$* (a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$\begin{aligned} &= \int (u^2 + 2u + 1) u^{1/2} du \\ &= \int (u^{5/2} + 2u^{3/2} + u^{1/2}) du \end{aligned}$$

Remember that :

$$* \int u^r dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{u^{7/2}}{7/2} + 2 \frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} + C$$

$$= \boxed{\frac{2}{7} (x-1)^{7/2} + \frac{4}{5} (x-1)^{5/2} + \frac{2}{3} (x-1)^{3/2} + C}.$$

Example 12 :

Evaluate $\int \cos^3 x \, dx$.

Solution

$$\int \cos^3 x \, dx$$

$$* \int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1$$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

* We can then evaluate the integral by substituting

$$\boxed{u = \sin x}$$

$$du = \cos x \, dx$$

* Therefore ,

$$\int \cos^3 x \, dx = \int (1 - u^2) \, du$$

Remember that :

$$* \int u^r \, dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= u - \frac{u^3}{3} + C$$

$$= \boxed{\sin x - \frac{1}{3} \sin^3 x + C}.$$

* The generalizations of Formula 12 , 13 , and 14 in Table 5.2.1 for $a > 0$ are

$$\boxed{\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C} \quad (5)$$

$$\int \frac{1}{\sqrt{a^2 - u^2}} du = \sin^{-1} \frac{u}{a} + C ; |u| < |a| \quad (6)$$

$$\int \frac{1}{u \sqrt{u^2 - a^2}} du = \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + C ; |u| > |a| \quad (7)$$

Example 13 :

Evaluate $\int \frac{dx}{a^2 + x^2}.$

Solution

$$\int \frac{dx}{a^2 + x^2}$$

* Applying (5) yields

$$\int \frac{dx}{a^2 + x^2} = \boxed{\frac{1}{a} \tan^{-1} \frac{x}{a} + C}.$$

Remember that :

$$* \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

Example 14 :

Evaluate $\int \frac{dx}{\sqrt{2 - x^2}}.$

Solution

$$\int \frac{dx}{\sqrt{2 - x^2}} = \int \frac{1}{\sqrt{(\sqrt{2})^2 - x^2}} dx$$

* Applying (6) with $\boxed{u = x}$ and $a = \sqrt{2}$ yields

$$\int \frac{dx}{\sqrt{2-x^2}} = \boxed{\sin^{-1} \frac{x}{\sqrt{2}} + C}.$$

Remember that :

$$* \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + C$$

EXERCISE SET 5.3: (Home Work)

16,17,29,31,33, Evaluate the following integrals using
35,39,41,45,46 appropriate substitutions.

$$16. \int x^3 \sqrt{5+x^4} dx$$

$$17. \int \sin 7x dx$$

$$29. \int \frac{x^3}{(5x^4+2)^3} dx$$

$$31. \int e^{\sin x} \cos x dx$$

$$33. \int x^2 e^{-2x^3} dx$$

$$35. \int \frac{e^x}{1+e^{2x}} dx$$

$$39. \int \cos^4 3t \sin 3t dx$$

$$41. \int x \sec^2(x^2) dx$$

$$45. \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx$$

$$46. \int \frac{\sin \theta}{\cos^2 \theta + 1} d\theta$$

5.5 THE DEFINITE INTEGRAL :

5.5.1 Definition :

A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

exist and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x) dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

which is called the definite integral of f from a to b . The numbers a and b are called the lower limit of integration and the upper limit of integration, respectively, and $f(x)$ is called the integrand.

5.5.2 Theorem :

If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is

$$A = \int_a^b f(x) dx \quad (1)$$

Example 1 :

Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

(a) $\int_1^4 2 dx$

(b) $\int_{-1}^2 (x + 2) dx$

(c) $\int_0^1 \sqrt{1-x^2} dx$

Solution

(a) $\int_1^4 2 dx$

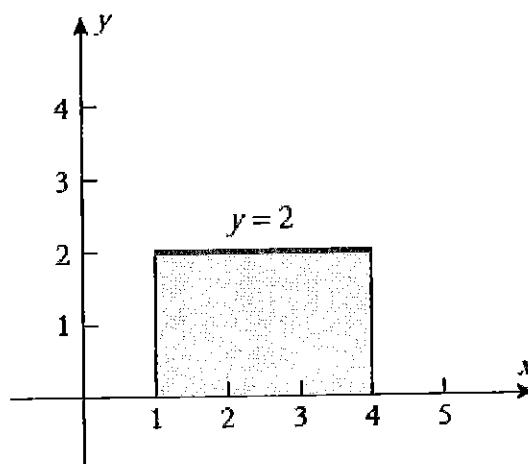


Figure 5.5.4a

* The graph of the integrand is the horizontal line $y = 2$, so the region is a rectangle of width 2 extending over the interval from 1 to 4 (Figure 5.5.4a). Thus,

$$\begin{aligned}\int_1^4 2 \, dx &= (\text{area of rectangle}) \\ &= (4 - 1)2 = (3)2 = \boxed{6}.\end{aligned}$$

Remember that :

* Area of a rectangle $A = \text{length} \times \text{width}$

(b) $\int_{-1}^2 (x + 2) \, dx$

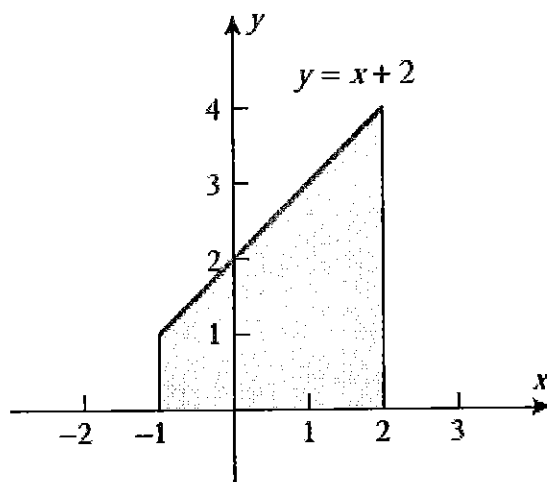


Figure 5.5.4b

* The graph of the integrand is the line $y = x + 2$, so the region is a trapezoid whose altitude extends from $x = -1$ to $x = 2$ (Figure 5.5.4b). Thus,

$$\begin{aligned}\int_{-1}^2 (x + 2) \, dx &= (\text{area of trapezoid}) \\ &= \frac{1}{2}(1 + 4)(2 - (-1))\end{aligned}$$

Remember that :

Area of a trapezoid $A = \frac{1}{2}(b_1 + b_2)h$

* b_1 and b_2 : lengths of parallel sides

h : distance between parallel sides (or altitude)

$$= \frac{1}{2}(5)(3) = \boxed{\frac{15}{2}}.$$

$$(c) \int_0^1 \sqrt{1-x^2} dx$$

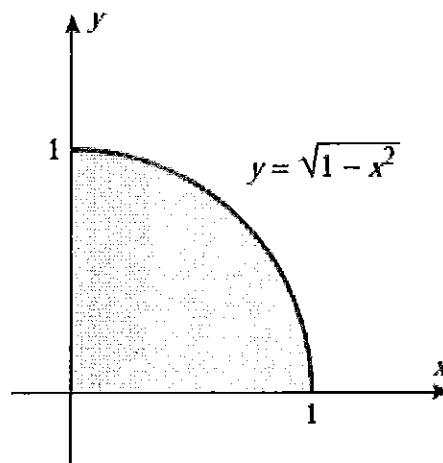


Figure 5.5.4c

* The graph of $y = \sqrt{1-x^2}$ is the upper semicircle of radius 1, centered at the origin, so the region is the right quarter-circle extending from $x = 0$ to $x = 1$ (Figure 5.5.4c). Thus,

$$\int_0^1 \sqrt{1-x^2} dx = (\text{area of quarter-circle})$$

$$= \frac{1}{4}\pi(1)^2 = \boxed{\frac{\pi}{4}}.$$

Remember that :

* *Area of a circle $A = \pi r^2$
 r : radius of circle*

Properties of the Definite Integral :

5.5.3 Definition :

(a) *If a is in the domain of f , we define*

$$\int_a^a f(x) dx = 0$$

(b) *If f is integrable on $[a, b]$, then we define*

$$\int_b^a f(x) dx = - \int_a^b f(x) dx$$

* *Part (a) of this definition is consistent with the intuitive idea that the area between a point on the x -axis and a curve $y = f(x)$ should be zero (Figure 5.5.6).*

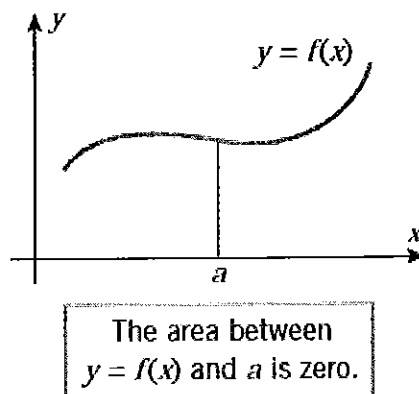


Figure 5.5.6

* Part (b) of the definition is simply a useful convention that interchanging the limits of integration reverses the sign of the integral.

Example 3 :

$$(a) \int_1^1 x^2 dx = \boxed{0}.$$

Remember that :

$$* \int_a^a f(x) dx = 0$$

$$(b) \int_1^0 \sqrt{1-x^2} dx = - \int_0^1 \sqrt{1-x^2} dx$$

Remember that :

$$* \int_b^a f(x) dx = - \int_a^b f(x) dx$$

$$= \boxed{-\frac{\pi}{4}}.$$

See Example 1(c)

5.5.4 Theorem :

If f and g are integrable on $[a, b]$ and if c is a constant, then cf , $f + g$, and $f - g$ are integrable on $[a, b]$ and

$$(a) \int_a^b cf(x) dx = c \int_a^b f(x) dx$$

$$(b) \int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

$$(c) \int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

* If f is continuous and nonnegative on the interval $[a, b]$, and if c is a point between a and b , then the area under $y = f(x)$ over the interval $[a, b]$ can be split into two parts and expressed as the area under the graph from a to c plus the area under the graph from c to b (Figure 5.5.7),

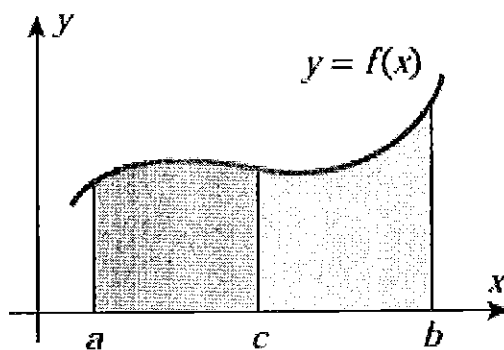


Figure 5.5.7

that is ,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

5.5.5 Theorem :

If f is integrable on a closed interval containing the three points a , b , and c , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

no matter how the points are ordered.

** The following theorem can also be motivated by interpreting definite integrals as areas.*

5.5.6 Theorem :

(a) If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x) dx \geq 0 .$$

(b) If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x) dx \geq \int_a^b g(x) dx .$$

Example 4 :

Evaluate $\int_0^1 (5 - 3\sqrt{1-x^2}) dx$.

Solution

** From parts (a) and (c) of Theorem 5.5.4 we can write*

$$\begin{aligned} \int_0^1 (5 - 3\sqrt{1-x^2}) dx &= \int_0^1 5 dx - \int_0^1 3\sqrt{1-x^2} dx \\ &= \int_0^1 5 dx - 3 \int_0^1 \sqrt{1-x^2} dx \end{aligned}$$

Remember that :

$$* \int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

$$* \int_a^b c f(x) dx = c \int_a^b f(x) dx$$

* The first integral in this difference can be interpreted as the area of a rectangle of length 5 and width 1, so its value is $\boxed{5}$, and from Example 1(c) the value of the second integral is $\boxed{\pi/4}$.

Remember that :

* Area of a rectangle $A = \text{length} \times \text{width}$

* Area of a circle $A = \pi r^2$

r : radius of circle

$$\int_0^1 \left(5 - 3\sqrt{1-x^2} \right) dx = 5 - 3 \left(\frac{\pi}{4} \right) = \boxed{5 - \frac{3\pi}{4}}.$$

EXERCISE SET 5.5: (Home Work)

13(a). Sketch the region whose signed area is represented by the definite integral $\int_0^3 x dx$, and evaluate the integral using an appropriate formula from geometry, where needed.

25. Use Theorem 5.5.4 and appropriate formula from geometry to evaluate the integral $\int_{-1}^3 (4 - 5x) dx$.

5.6 THE FUNDAMENTAL THEOREM OF CALCULUS :

The Fundamental Theorem of Calculus :

* As in earlier sections , let us begin by assuming that f is nonnegative and continuous on an interval $[a,b]$, in which case the area A under the graph of f over the interval $[a,b]$ is represented by the definite integral

$$A = \int_a^b f(x) dx \quad (1)$$

(Figure 5.6.1).

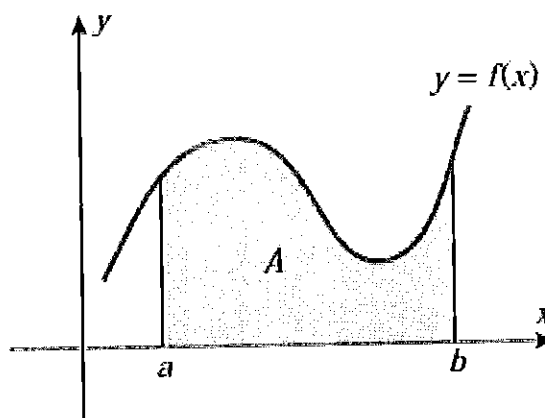


Figure 5.6.1

5.6.1 Theorem (The Fundamental Theorem of Calculus , Part 1) :

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (2)$$

Proof

* Let $x_1, x_2, \dots, x_{n-1}$ be any point in $[a, b]$ such that

$$a < x_1 < x_2 < \dots < x_{n-1} < b$$

* These values divide $[a, b]$ into n subintervals

$$[a, x_1], [x_1, x_2], \dots, [x_{n-1}, b] \quad (3)$$

whose lengths, as usual, we denote by

$$\Delta x_1, \Delta x_2, \dots, \Delta x_n$$

(see Figure 5.6.3).

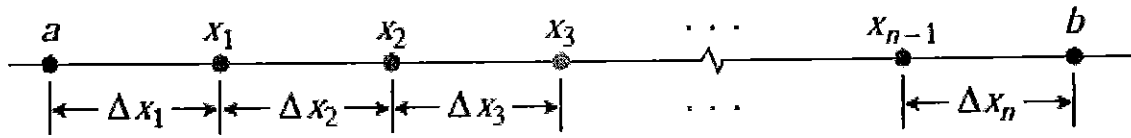


Figure 5.6.3

* By hypothesis, $F'(x) = f(x)$ for all x in $[a, b]$, so F satisfies the hypotheses of the Mean-Value Theorem (4.8.2) on each subinterval in (3). Hence, we can find points $x_1^*, x_2^*, \dots, x_n^*$ in the respective subintervals in (3) such that

$$\begin{aligned} F(x_1) - F(a) &= F'(x_1^*)(x_1 - a) = f(x_1^*)\Delta x_1 \\ F(x_2) - F(x_1) &= F'(x_2^*)(x_2 - x_1) = f(x_2^*)\Delta x_2 \\ F(x_3) - F(x_2) &= F'(x_3^*)(x_3 - x_2) = f(x_3^*)\Delta x_3 \\ &\vdots \\ F(b) - F(x_{n-1}) &= F'(x_n^*)(b - x_{n-1}) = f(x_n^*)\Delta x_n \end{aligned}$$

* Adding the preceding equations yields

$$F(b) - F(a) = \sum_{k=1}^n f(x_k^*) \Delta x_k \quad (4)$$

* Let us now increase n in such a way that $\max \Delta x_k \rightarrow 0$. Since f is assumed to be continuous, the right side of (4) approaches $\int_a^b f(x) dx$ by Theorem 5.5.2 and Definition 5.5.1.

However, the left side of (4) is independent of n ; that is, the left side of (4) remains constant as n increases. Thus,

$$F(b) - F(a) = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k = \int_a^b f(x) dx$$

* We can write Theorem 5.6.1 as

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a) \quad (5)$$

* In words, Theorem 5.4.1 states :

The definite integral can be evaluated by any antiderivative of the integrand and then subtracting the value of this antiderivative at the lower limit of integration from its value at the upper limit of integration.

Example 1 :

Evaluate $\int_1^2 x dx$.

Solution

* The function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$; thus from (2)

$$\int_1^2 x \, dx = \left[\frac{1}{2} x^2 \right]_1^2 = \frac{1}{2} \left[(2)^2 - (1)^2 \right]$$

Remember that :

$$* \int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$* \int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a)$$

$$= \frac{1}{2} (4 - 1) = \boxed{\frac{3}{2}}.$$

Example 2 :

Evaluate $\int_0^3 (9 - x^2) \, dx$.

Solution

* Using the Fundamental Theorem of Calculus :

$$\begin{aligned} \int_0^3 (9 - x^2) \, dx &= \left[9x - \frac{x^3}{3} \right]_0^3 = \left(9(3) - \frac{(3)^3}{3} \right) - 0 \\ &= 27 - 9 = \boxed{18}. \end{aligned}$$

Remember that :

$$* \int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$* \int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a)$$

Example 3 :

- (a) Find the area under the curve $y = \cos x$ over the interval $[0, \pi/2]$ (Figure 5.6.4).

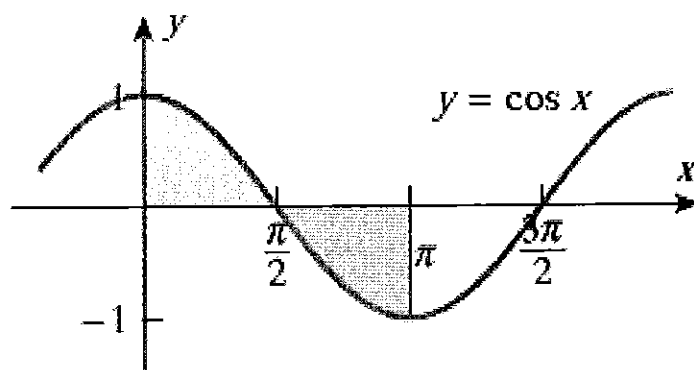


Figure 5.6.4

- (b) Make a conjecture about the value of the integral

$$\int_0^{\pi} \cos x \, dx$$

and confirm your conjecture using the Fundamental Theorem of Calculus.

Solution

- (a) Since $\cos x \geq 0$ over the interval $[0, \pi/2]$, the area A under the curve is

$$A = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2}$$

Remember that :

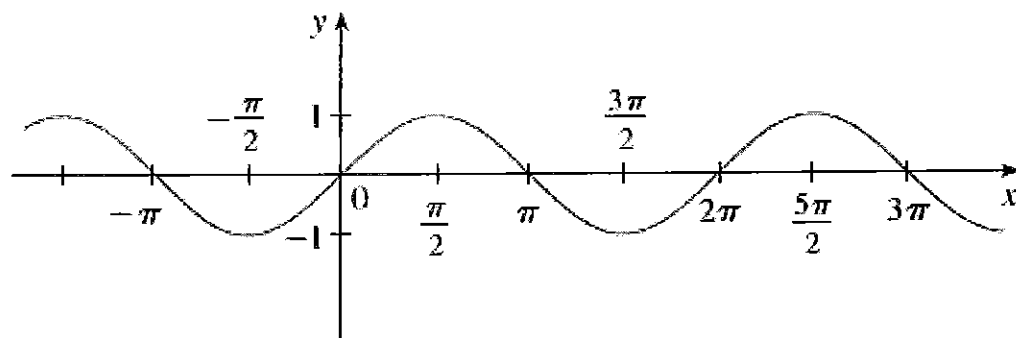
$$* \int \cos x \, dx = \sin x + C$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = \boxed{1}.$$

Remember that :

$$* \sin 0 = 0, \sin \frac{\pi}{2} = 1$$

$$y = \sin x$$



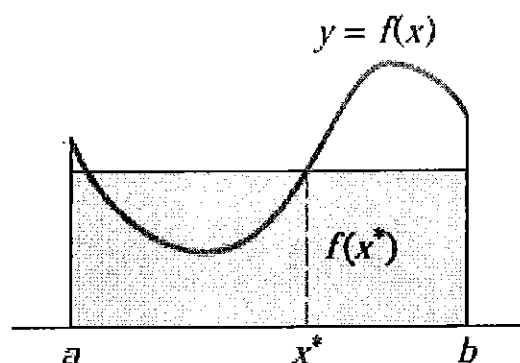
- (b) The given integral can be interpreted as the signed area between the graph of $y = \cos x$ and the interval $[0, \pi]$. The graph in Figure 5.6.4 suggests that over the interval $[0, \pi]$ the portion of area above the x-axis is the same as the portion of area below the x-axis, so we conjecture that the signed area is **zero**; this implies that the value of the integral is zero. This is confirmed by the computations

$$\int_0^{\pi} \cos x \, dx = [\sin x]_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = \mathbf{0}.$$

Remember that :

$$* \sin 0 = 0, \sin \pi = 0$$

The Mean-Value Theorem for Integrals :



The area of the shaded rectangle is equal to the area of the shaded region in Figure 5.6.9.

Figure 5.6.10

5.6.2 Theorem (The Mean-Value Theorem for Integrals) :

If f is continuous on a closed interval $[a, b]$, then there is at least one point x^* in $[a, b]$ such that

$$\int_a^b f(x) dx = f(x^*)(b - a) \quad (8)$$

Example 9 :

* Since $f(x) = x^2$ is continuous on the interval $[1, 4]$, the Mean-Value Theorem for Integrals guarantees that there is a point x^* in $[1, 4]$ such that

$$\int_1^4 x^2 dx = f(x^*)(4-1) = (x^*)^2(4-1) = 3(x^*)^2$$

* But

$$\int_1^4 x^2 dx = \frac{1}{3} [x^3]_1^4 = \frac{1}{3} [(4)^3 - (1)^3] = \frac{1}{3} (64 - 1) = \boxed{21}.$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

so that

$$3(x^*)^2 = 21 \quad \text{or} \quad (x^*)^2 = 7 \quad \text{or} \quad x^* = \pm\sqrt{7}$$

* Thus, $x^* = \sqrt{7} \approx 2.65$ is the point in the interval $[1, 4]$ whose existence is guaranteed by the Mean-Value Theorem for Integrals.

EXERCISE SET 5.6:

(Home Work)

13,17,19,20,21, Evaluate the integrals using Part 1 of the
22,23,24,25,26 Fundamental Theorem of Calculus.

$$13. \int_{-2}^1 (x^2 - 6x + 12) dx$$

$$17. \int_4^9 2x\sqrt{x} dx$$

$$19. \int_{-\pi/2}^{\pi/2} \sin \theta d\theta$$

$$20. \int_0^{\pi/4} \sec^2 \theta d\theta$$

$$21. \int_{-\pi/4}^{\pi/4} \cos \theta d\theta$$

$$22. \int_0^{\pi/3} (2x - \sec x \tan x) dx$$

5.9 EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION:

Two Methods for Making Substitutions in Definite Integrals :

* Recall from Section 5.3 that indefinite integrals of the form

$$\int f(g(x)) g'(x) dx$$

can sometimes be evaluated by making the u -substitution

$$\boxed{u = g(x)} \quad , \quad \boxed{du = g'(x) dx} \quad (1)$$

which converts the integral to the form

$$\int f(u) du$$

* To apply this method to a definite integral of the form

$$\int_a^b f(g(x)) g'(x) dx$$

we need to account for the effect that the substitution has on the x -limit of integration.

* There are two ways of doing this.

Method 1 :

* First evaluate the indefinite integral

$$\int f(g(x)) g'(x) dx$$

by substitution , and then use the relationship

$$\int_a^b f(g(x)) g'(x) dx = \left[\int f(g(x)) g'(x) dx \right]_a^b$$

to evaluate the definite integral. This procedure does not require any modification of the x -limits of integration.

Method 2 :

* *Make the substitution (1) directly in the definite integral , and then use the relationship $\boxed{u = g(x)}$ to replace the x -limits , $x = a$ and $x = b$, by corresponding u -limits , $u = g(a)$ and $u = g(b)$. This produces a new definite integral*

$$\int_{g(a)}^{g(b)} f(u) du$$

that is expressed entirely in terms of u .

Example 1 :

Use the two methods above to evaluate $\int_0^2 x(x^2 + 1)^3 dx$.

Solution

$$\int_0^2 x(x^2 + 1)^3 dx$$

Method 1 :

* *If we let*

$$\boxed{u = x^2 + 1} \text{ so that } du = 2x dx \left(\text{or } \frac{1}{2} du = x dx \right)$$

Then we obtain

$$\int x(x^2 + 1)^3 dx = \frac{1}{2} \int u^3 du$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{1}{2} \frac{u^4}{4} + C = \frac{1}{8} (x^2 + 1)^4 + C$$

* Thus ,

$$\begin{aligned} \int_0^2 x (x^2 + 1)^3 dx &= \left[\int x (x^2 + 1)^3 dx \right]_{x=0}^2 \\ &= \left[\frac{1}{8} (x^2 + 1)^4 \right]_0^2 = \frac{1}{8} \left[((2)^2 + 1)^4 - ((0)^2 + 1)^4 \right] \\ &= \frac{1}{8} [625 - 1] = \frac{624}{8} = \boxed{78}. \end{aligned}$$

Method 2 :

* If we let

$$\boxed{u = x^2 + 1} \text{ so that } du = 2x dx \left(\text{or } \frac{1}{2} du = x dx \right)$$

Then

$$\text{if } x = 0, \quad u = (0)^2 + 1 = 1$$

$$\text{if } x = 2, \quad u = (2)^2 + 1 = 5$$

* Thus ,

$$\int_0^2 x (x^2 + 1)^3 dx = \frac{1}{2} \int_1^5 u^3 du$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{1}{2} \left[\frac{u^4}{4} \right]_{u=1}^5 = \frac{1}{8} [(5)^4 - (1)^4]$$

$$= \frac{1}{8} [625 - 1] = \frac{624}{8} = \boxed{78}.$$

which agrees with the result obtained by Method 1.

* The following theorem states precise conditions under which Method 2 can be used.

5.9.1 Theorem :

If g' is continuous on $[a, b]$ and f is continuous on an interval containing the values of $g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Example 2 :

Evaluate

$$(a) \int_0^{\pi/8} \sin^5 2x \cos 2x dx \quad (b) \int_2^5 (2x-5)(x-3)^9 dx$$

Solution

$$(a) \int_0^{\pi/8} \sin^5 2x \cos 2x dx = \int_0^{\pi/8} (\sin 2x)^5 \cos 2x dx$$

* If we let

$$\boxed{u = \sin 2x} \quad \text{so that} \quad du = 2 \cos 2x dx$$

$$\text{or} \quad \frac{1}{2} du = \cos 2x dx$$

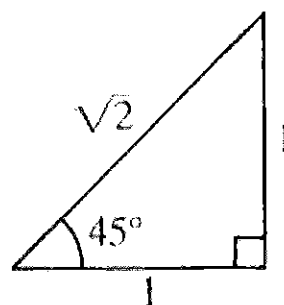
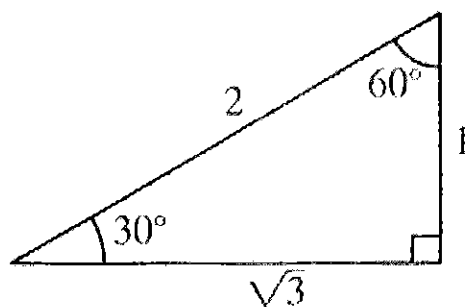
With this substitution ,

$$\text{if } x = 0 \quad , \quad u = \sin(0) = 0$$

$$\text{if } x = \pi/8 \quad , \quad u = \sin(\pi/4) = 1/\sqrt{2}$$

Remember that :

$$* \quad \frac{\pi}{4} = \frac{\pi}{4} \frac{180^\circ}{\pi} = 45^\circ \quad , \quad \sin 0 = 0 \quad , \quad \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$



* Thus ,

$$\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx = \frac{1}{2} \int_0^{1/\sqrt{2}} u^5 \, du$$

Remember that :

$$* \quad \int u^r \, dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{1}{2} \left[\frac{u^6}{6} \right]_{u=0}^{1/\sqrt{2}} = \frac{1}{12} \left[\left(\frac{1}{\sqrt{2}} \right)^6 - (0)^6 \right]$$

$$= \frac{1}{12} \left(\frac{1}{8} \right) = \boxed{\frac{1}{96}}.$$

$$(b) \int_2^5 (2x-5)(x-3)^9 \, dx$$

* If we let

$$\boxed{u = x - 3} \quad \text{so that} \quad du = dx$$

This leaves a factor of $2x - 5$ unresolved in the integrand.
However

$$x = u + 3, \quad \text{so} \quad 2x - 5 = 2(u + 3) - 5 = 2u + 1$$

With this substitution,

$$\text{if } x = 2, \quad u = 2 - 3 = -1$$

$$\text{if } x = 5, \quad u = 5 - 3 = 2$$

* Thus,

$$\begin{aligned} \int_2^5 (2x - 5)(x - 3)^9 dx &= \int_{-1}^2 (2u + 1)u^9 du \\ &= \int_{-1}^2 (2u^{10} + u^9) du \end{aligned}$$

Remember that :

$$* \int u^r dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$\begin{aligned} &= \left[\frac{2u^{11}}{11} + \frac{u^{10}}{10} \right]_{u=-1}^2 \\ &= \left(\frac{2(2)^{11}}{11} + \frac{(2)^{10}}{10} \right) - \left(\frac{2(-1)^{11}}{11} + \frac{(-1)^{10}}{10} \right) \\ &= \boxed{\frac{52,233}{110} \approx 474.8} . \end{aligned}$$

Example 3 :

Evaluate

$$(a) \int_0^{3/4} \frac{dx}{1-x}$$

$$(b) \int_0^{\ln 3} e^x (1 + e^x)^{1/2} dx$$

Solution

$$(a) \int_0^{3/4} \frac{dx}{1-x}$$

* If we let

$$\boxed{u = 1 - x} \quad \text{so that} \quad du = -dx \\ \text{or} \quad -du = dx$$

With this substitution ,

$$\text{if } x = 0, \quad u = 1 - 0 = 1$$

$$\text{if } x = 3/4, \quad u = 1 - 3/4 = 1/4$$

* Thus ,

$$\int_0^{3/4} \frac{dx}{1-x} = - \int_1^{1/4} \frac{1}{u} du$$

Remember that :

$$* \int \frac{1}{u} du = \ln|u| + C$$

$$\begin{aligned} &= - \left[\ln|u| \right]_{u=1}^{1/4} = - \left[\ln \left| \frac{1}{4} \right| - \ln|1| \right] \\ &= - \left[\ln|1| - \ln|4| - \ln|1| \right] = \boxed{\ln 4} . \end{aligned}$$

Remember that :

$$* \text{ If } a > 0, b > 0, \text{ then } \ln\left(\frac{a}{b}\right) = \ln a - \ln b, \quad \ln 1 = 0$$

$$(b) \int_0^{\ln 3} e^x (1 + e^x)^{1/2} dx$$

* If we let

$$\boxed{u = 1 + e^x} \quad \text{so that} \quad du = e^x dx$$

With this substitution ,

$$\text{if } x = 0 \quad , \quad u = 1 + e^0 = 1 + 1 = 2$$

$$\text{if } x = \ln 3 \quad , \quad u = 1 + e^{\ln 3} = 1 + 3 = 4$$

Remember that :

$$* \quad e^0 = 1 \quad , \quad e^{\ln x} = x \quad , \quad x > 0$$

* Thus ,

$$\int_0^{\ln 3} e^x (1 + e^x)^{1/2} dx = \int_2^4 u^{1/2} du$$

Remember that :

$$* \quad \int u^r dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$\begin{aligned} &= \left[\frac{u^{3/2}}{3/2} \right]_{u=2}^4 = \frac{2}{3} \left[(4)^{3/2} - (2)^{3/2} \right] \\ &= \frac{2}{3} \left[2^3 - 2^{3/2} \right] = \boxed{\frac{4}{3} (4 - \sqrt{2})} . \end{aligned}$$

EXERCISE SET 5.9: (Home Work)

5,11,12,13,16 Evaluate the definite integral two ways : first by a u -substitution in the definite integral and then by a u -substitution in the corresponding indefinite integral.

$$5. \quad \int_0^1 (2x + 1)^3 dx$$

$$11. \quad \int_0^{\pi/2} 4 \sin(x/2) dx$$

$$12. \int_0^{\pi/6} 2 \cos 3x \, dx$$

$$13. \int_{-2}^{-1} \frac{x}{(x+2)^3} \, dx$$

$$16. \int_0^{\ln 5} e^x (3 - 4e^x) \, dx$$

32,37,38,46,47 Evaluate the integrals by any method.

$$32. \int_1^2 \sqrt{5x-1} \, dx$$

$$37. \int_0^{\pi/4} 4 \sin x \cos x \, dx$$

$$38. \int_0^{\pi/4} \sqrt{\tan x} \sec^2 x \, dx$$

$$46. \int_0^{\sqrt{2}} x e^{-x^2} \, dx$$

$$47. \int_0^1 \frac{x}{\sqrt{4-3x^4}} \, dx$$

5.10 LOGARITHMIC AND OTHER FUNCTIONS DEFINED BY INTEGRALS :

The Connection Between Natural Logarithms and Integrals :

5.10.1 Definition :

The natural logarithm of x is denoted by $\ln x$ and is defined by the integral

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0 \quad (1)$$

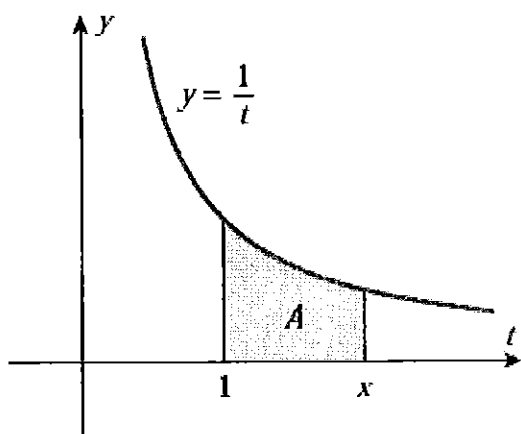


Figure 5.10.2

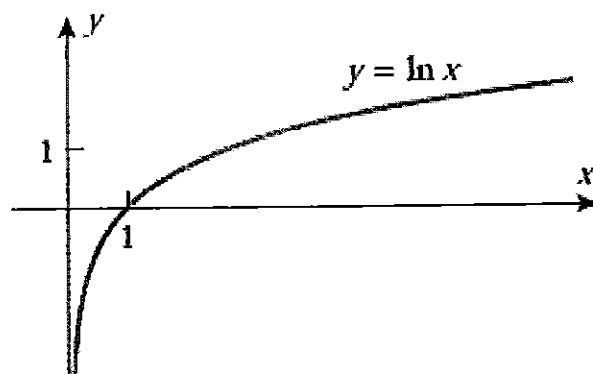


Figure 5.10.3

Algebraic Properties of $\ln x$:

5.10.2 Theorem :

For any positive numbers a and c and any rational number r :

(a) $\ln ac = \ln a + \ln c$

(b) $\ln \frac{1}{c} = -\ln c$

(c) $\ln \frac{a}{c} = \ln a - \ln c$

(d) $\ln a^r = r \ln a$

Remember that :

$$* \ln 1 = 0$$

Domain , Range , and End Behavior of $\ln x$:

5.10.3 Theorem :

(a) The domain of $\ln x$ is $(0, +\infty)$.

(b) $\lim_{x \rightarrow 0^+} \ln x = -\infty$ and $\lim_{x \rightarrow +\infty} \ln x = +\infty$.

(c) The range of $\ln x$ is $(-\infty, +\infty)$.

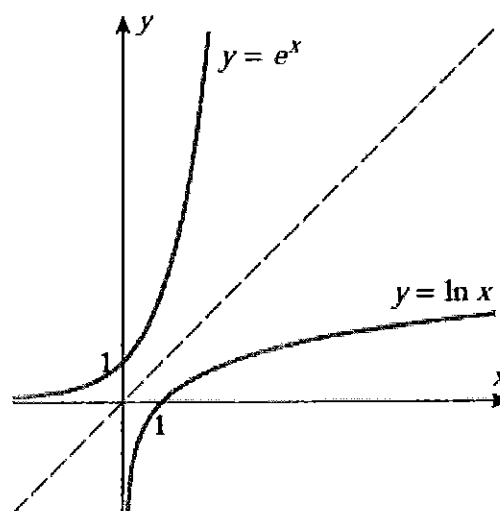
Definition of e^x :

5.10.4 Definition :

The inverse of the natural logarithm function $\ln x$ is defined by e^x and is called the natural exponential function.

Remember that :

$$* \lim_{x \rightarrow -\infty} e^x = 0, \quad e^0 = 1, \quad \lim_{x \rightarrow +\infty} e^x = +\infty$$



$$* \quad f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

$$* \quad \ln e^x = x, \quad x \in \mathbb{R}, \quad e^{\ln x} = x, \quad x > 0$$

5.10.5 Theorem :

The natural exponential function e^x is differentiable, and hence continuous, on $(-\infty, +\infty)$, and its derivative is

$$\frac{d}{dx} [e^x] = e^x$$

Irrational Exponents :

5.10.6 Definition :

If $a > 0$ and r is a real number, a^r is defined by

$$a^r = e^{r \ln a} \quad (7)$$

Remember that :

If $a > 0$ and r is real number, then

$$a^r = e^{\ln a^r} = e^{r \ln a}$$

** With this definition it can be shown that the standard algebraic properties of exponents, such as*

$$a^p a^q = a^{p+q}, \quad \frac{a^p}{a^q} = a^{p-q}, \quad (a^p)^q = a^{pq}, \quad (a^p)(b^p) = (ab)^p$$

hold for any real values of a, b, p , and q , where a and b are positive.

5.10.7 Theorem :

(a) For any real number r , the power function x^r is differentiable on $(0, +\infty)$ and its derivative is

$$\frac{d}{dx} [x^r] = r x^{r-1}$$

(b) For $b > 0$ and $b \neq 1$, the base b exponential function b^x is differentiable on $(-\infty, +\infty)$ and its derivative is

$$\frac{d}{dx} [b^x] = b^x \ln b$$

5.10.8 Theorem :

$$(a) \lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

$$(b) \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$(c) \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

General Logarithms :

* We note that for $b > 0$ and $b \neq 1$, the function b^x is one-to-one and so has an inverse function. Using the definition of b^x , we can solve $y = b^x$ for x as a function of y :

$$y = b^x = e^{x \ln b}$$

$$\ln y = \ln(e^{x \ln b}) = x \ln b$$

$$\frac{\ln y}{\ln b} = x$$

Thus, the inverse function for b^x is $(\ln x) / (\ln b)$.

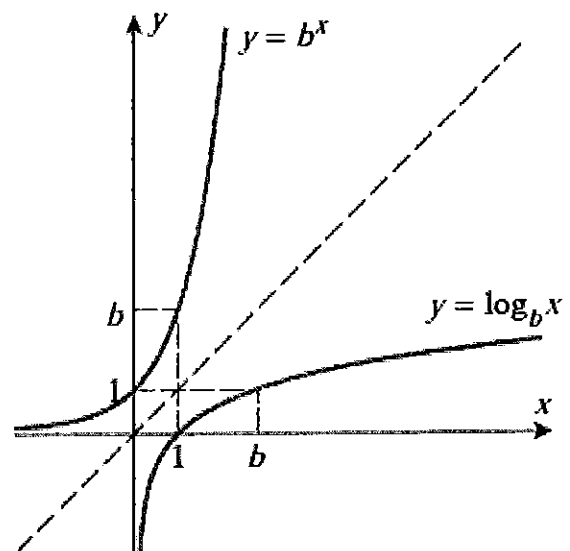
5.10.9 Definition :

For $b > 0$ and $b \neq 1$, the base b logarithm function, denoted $\log_b x$, is defined by

$$\log_b x = \frac{\ln x}{\ln b} \quad (9)$$

* It follows immediately from this definition that $\log_b x$ is the inverse function for b^x and satisfies the properties in Table 0.5.3.

Remember that :



* Furthermore, $\log_b x$ is differentiable, and hence continuous, on $(0, +\infty)$, and its derivative is

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}$$

Remember that :

$$* \frac{d}{dx}[\ln x] = \frac{1}{x}$$

* *As a final of consistency, we observe that $\log_e x = \ln x$.*

Remember that :

*

$\log_e x = \ln x$ (natural logarithmic function),

$\log_{10} x = \log x$ (common logarithmic function)

CHAPTER (5)

APPLICATIONS OF THE DEFINITE INTEGRAL IN GEOMETRY, SCIENCE, AND ENGINEERING

6.1 AREA BETWEEN TWO CURVES :

A Review of Riemann Sums :

* Before we consider the problem of finding the area between two curves it will be helpful to review the basic principle that underlies the calculation of area as a definite integral. Recall that if f is continuous and nonnegative on $[a, b]$, then the definite integral for the area A under $y = f(x)$ over the interval $[a, b]$ is obtained in four steps (Figure 6.1.1) :

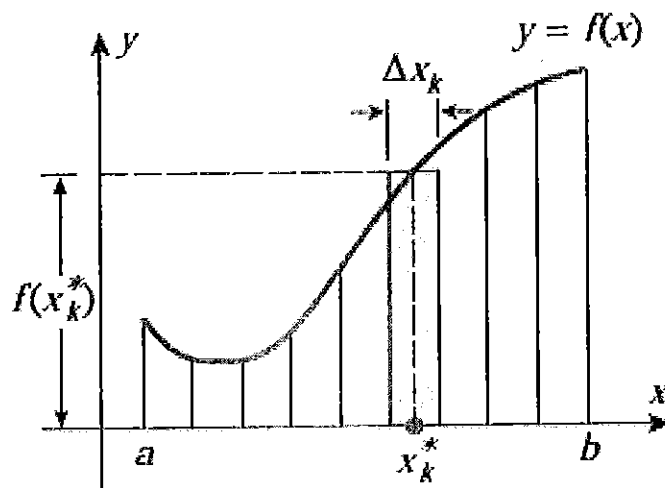


Figure 6.1.1

- Divide the interval $[a, b]$ into n subintervals, and divide the region under the curve $y = f(x)$ into n strips.
- Assuming that the width of the k th strip is Δx_k , approximate the area of that strip by the area $f(x_k^*) \Delta x_k$ of a rectangle

of width Δx_k and height $f(x_k^*)$, where x_k^* is a point in the k th subinterval.

- Add the approximate areas of the strips to approximate the entire area A by the Riemann sum :

$$A \approx \sum_{k=1}^n f(x_k^*) \Delta x_k$$

- Take the limit of the Riemann sums as the number of subintervals increases and all their widths approach zero. This causes the error in the approximations to approach zero and produces the following definite integral for the exact area A :

$$A = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k = \int_a^b f(x) dx$$

- * Figure 6.1.2 illustrates the effect that the limit process has on the various parts of the Riemann sum :

$$\begin{array}{ccc} \sum_{k=1}^n & f(x_k^*) & \Delta x_k \\ \downarrow & \downarrow & \downarrow \\ \int_a^b & f(x) & dx \end{array}$$

Effect of the limit process
on the Riemann sum

Figure 6.1.2

- The quantity x_k^* in the Riemann sum becomes the variable x in the definite integral.
- The interval width Δx_k in the Riemann sum becomes the dx in the definite integral.
- The interval $[a, b]$, which is the union of the subintervals with widths $\Delta x_1, \Delta x_2, \dots, \Delta x_n$, does not appear explicitly in the Riemann sum but is represented by the upper and lower limits of integration in the definite integral.

Area Between $y = f(x)$ and $y = g(x)$:

6.1.1 First Area Problem :

Suppose that f and g are continuous functions on an interval $[a, b]$ and

$$f(x) \geq g(x) \quad \text{for } a \leq x \leq b$$

[This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.1.3a).

* To solve this problem we divide the interval $[a, b]$ into n subintervals, which has the effect of subdividing the region into n strips (Figure 6.1.3b).

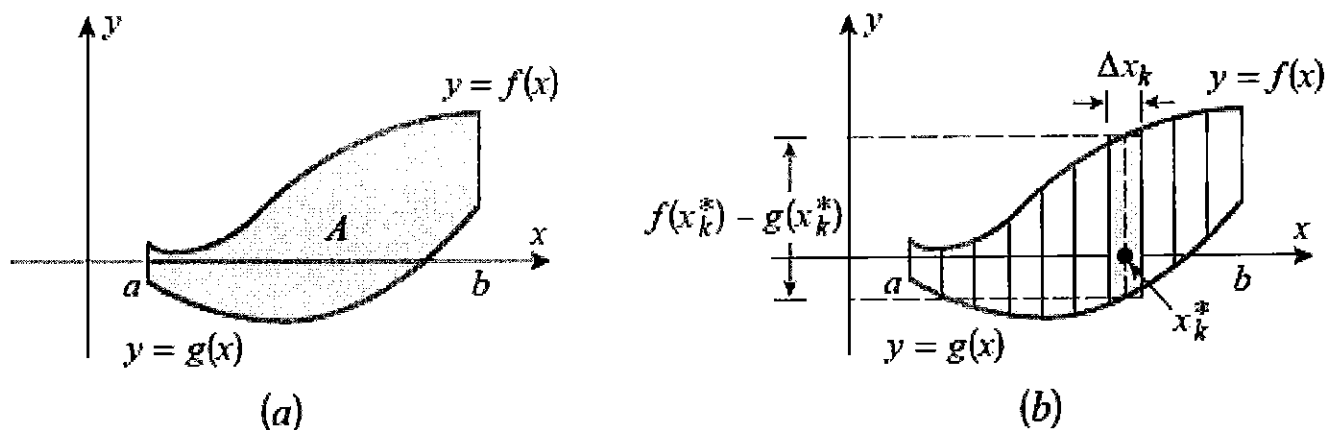


Figure 6.1.3

* If we assume that the width of the k th strip is Δx_k , then the area of the strip can be approximated by the area of a rectangle of width Δx_k and height $f(x_k^*) - g(x_k^*)$, where x_k^* is a point in the k th subinterval. Adding these approximations yields the following Riemann sum that approximates the area A :

$$A \approx \sum_{k=1}^n [f(x_k^*) - g(x_k^*)] \Delta x_k$$

* Taking the limit as n increases and the widths of all the subintervals approach zero yields the following definite integral for the area A between the curves :

$$\begin{aligned} A &= \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n [f(x_k^*) - g(x_k^*)] \Delta x_k \\ &= \int_a^b [f(x) - g(x)] dx \end{aligned}$$

* In summary, we have the following result.

6.1.2 Area Formula :

If f and g are continuous functions on an interval $[a, b]$ and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)] dx \quad (1)$$

Example 1 :

Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution

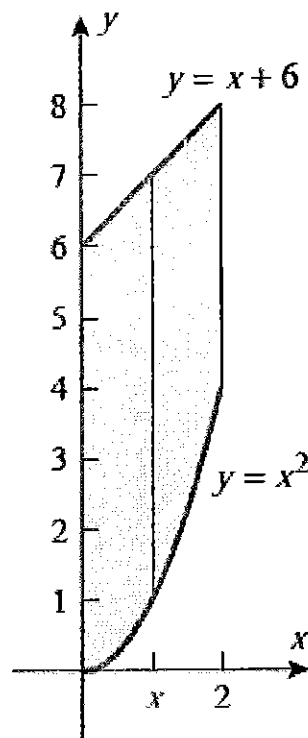


Figure 6.1.4

* The region and a cross section are shown in Figure 6.1.4. The cross section extends from $g(x) = x^2$ on the bottom to $f(x) = x + 6$ on the top. If the cross section is moved through the region, then its leftmost position will be $x = 0$ and its rightmost position will be $x = 2$. Thus, from (1)

$$A = \int_a^b [f(x) - g(x)] dx$$

$$= \int_0^2 [(x + 6) - (x^2)] dx$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

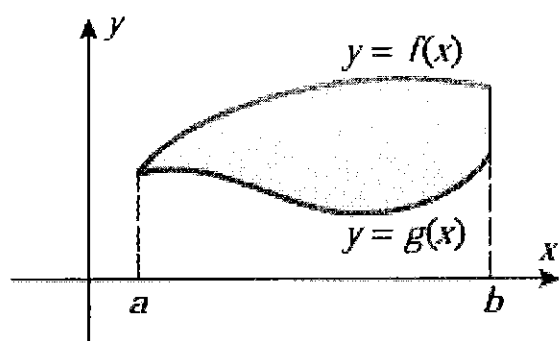
$$* \int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_0^2$$

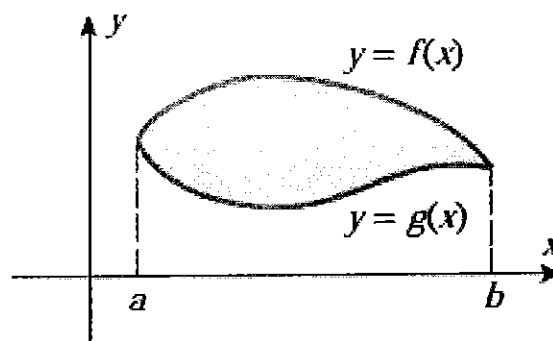
$$= \left[\frac{(2)^2}{2} + 6(2) - \frac{(2)^3}{3} \right] - [0]$$

$$= 2 + 12 - \frac{8}{3} = \boxed{\frac{34}{3}}.$$

* It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments (Figure 6.1.5). When that occurs you will have to determine the points of intersection to obtain the limits of integration.



The left-hand boundary reduces to a point.



Both side boundaries reduce to points.

Figure 6.1.5

Example 2 :

Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution

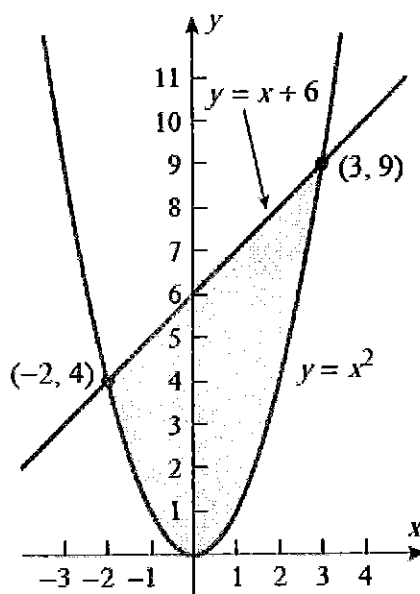


Figure 6.1.6

- * A sketch of the region (Figure 6.1.6) shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. At the endpoints of the region, the upper and lower

boundaries have the same y-coordinates ; thus , to find the endpoints we equate

$$y = x^2 \quad \text{and} \quad y = x + 6 \quad (2)$$

This yields

$$x^2 = x + 6$$

or $x^2 - x - 6 = 0$

or $(x + 2)(x - 3) = 0$

from which we obtain

$$\boxed{x = -2} \quad \text{and} \quad \boxed{x = 3}$$

Although the y-coordinates of the endpoints are not essential to our solution , they may be obtained from (2) by substituting $x = -2$ and $x = 3$ in either equation. This yields $y = 4$ and $y = 9$, so the upper and lower boundaries intersect at $\boxed{(-2, 4)}$ and $\boxed{(3, 9)}$.

* From (1) with $f(x) = x + 6$, $g(x) = x^2$, $a = -2$, and $b = 3$, we obtain the area

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)] dx \\ &= \int_{-2}^3 [(x + 6) - (x^2)] dx \end{aligned}$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^3$$

$$= \left[\frac{(3)^2}{2} + 6(3) - \frac{(3)^3}{3} \right] - \left[\frac{(-2)^2}{2} + 6(-2) - \frac{(-2)^3}{3} \right]$$

$$= \frac{27}{2} - \left(-\frac{22}{3} \right) = \boxed{\frac{125}{6}}.$$

* In the case where f and g are nonnegative on the interval $[a, b]$, the formula

$$A = \int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

states that the area A between the curves can be obtained by subtracting the area under $y = g(x)$ from the area under $y = f(x)$ (Figure 6.1.7).

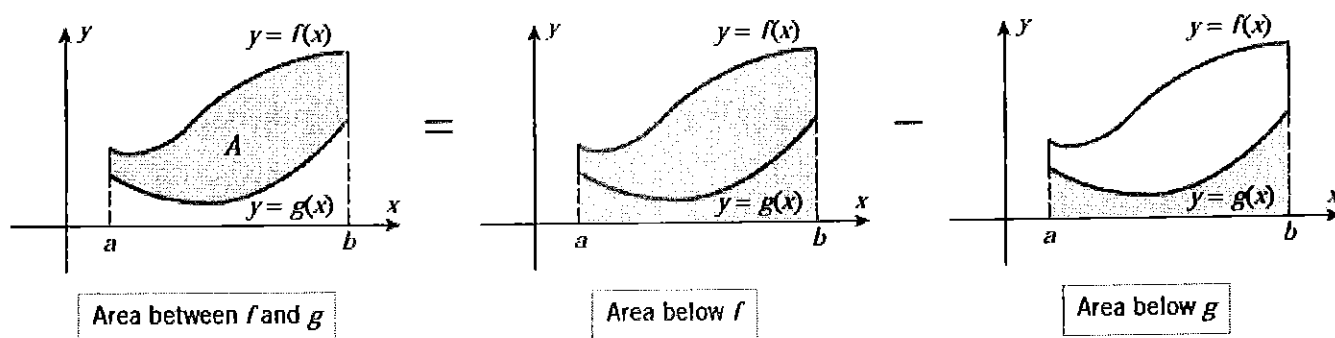
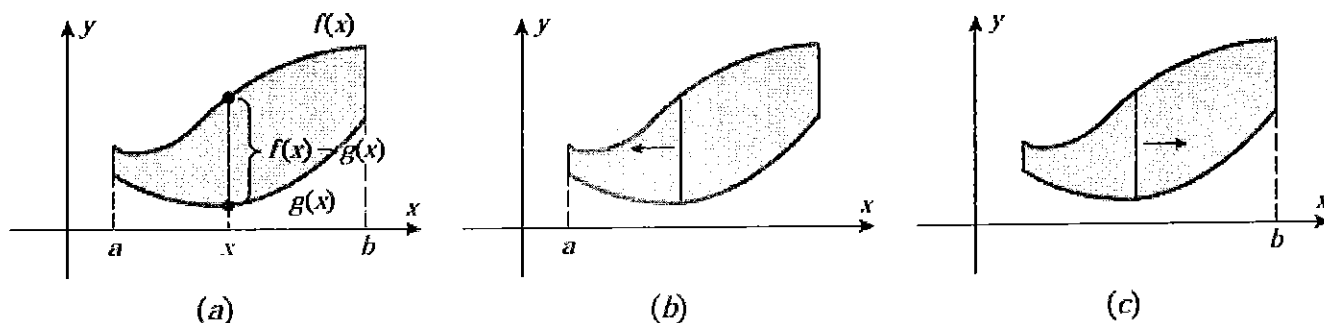


Figure 6.1.7

* Some regions may require careful thought to determine the integrand and limits of integration in (1). Here is a systematic procedure that you can follow to set up this formula.

Finding the Limits of Integration for the Area Between Two Curves :

- Step 1.** Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x -axis, connecting the top and bottom boundaries (Figure 6.1.9a).
- Step 2.** The y -coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).
- Step 3.** To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$ (Figures 6.1.9b and 6.1.9c).



Figures 6.1.9

* There is a useful way of thinking about this procedure :

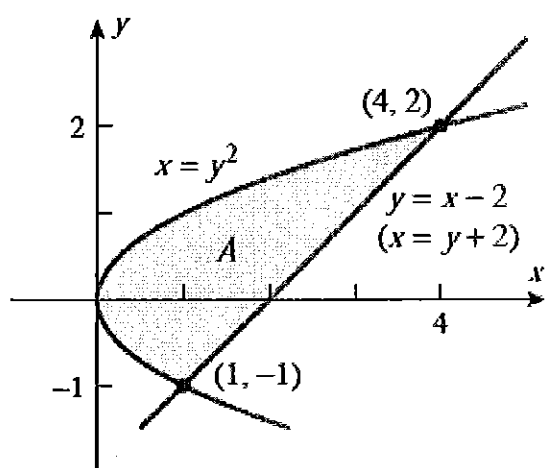
If you view the vertical line segment as the "cross section" of the region at the point x , then Formula (1) states that the area between the curves is obtained by integrating the length of the cross section over the interval $[a, b]$.

- * Note that it is not necessary to make an extremely accurate sketch in Step 1 ; the only purpose of the sketch is to determine which curve is the upper boundary and which is the lower boundary.
- * It is possible for the upper or lower boundary of a region to consist of two or more different curves , in which case it will be convenient to subdivide the region into smaller pieces in order to apply Formula (1) . This is illustrated in the next example.

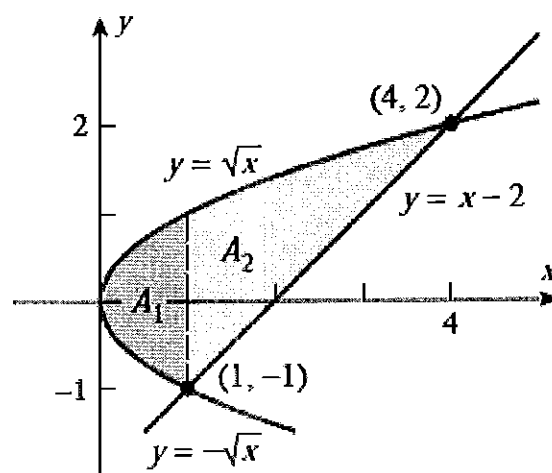
Example 4 :

Find the area of the region that is enclosed by $x = y^2$ and $y = x - 2$.

Solution



(a)



(b)

Figure 6.1.10

- * To determine the appropriate boundaries of the region , we need to know where the curves $x = y^2$ and $y = x - 2$ intersect. In Example 2 we found intersections by equating the expressions for y . Here it is easier to rewrite the latter

equation as $x = y + 2$ and equate the expressions for x , namely,

$$x = y^2 \quad \text{and} \quad x = y + 2 \quad (3)$$

This yields

$$y^2 = y + 2$$

$$\text{or} \quad y^2 - y - 2 = 0$$

$$\text{or} \quad (y + 1)(y - 2) = 0$$

from which we obtain $y = -1$ and $y = 2$.

Substituting these values in either equation in (3) we see that the corresponding x -values are $x = 1$ and $x = 4$, respectively, so the points of intersection are $(1, -1)$ and $(4, 2)$ (Figure 6.1.10a).

- * To apply Formula (1), the equations of the boundaries must be written so that y is expressed explicitly as a function of x . The upper boundary can be written as $y = \sqrt{x}$ (rewrite $x = y^2$ as $y = \pm\sqrt{x}$ and choose the $+$ for the upper portion of the curve). The lower boundary consists of two parts:

$$y = -\sqrt{x} \quad \text{for} \quad 0 \leq x \leq 1$$

$$\text{and} \quad y = x - 2 \quad \text{for} \quad 1 \leq x \leq 4$$

(Figure 6.1.10b). Because of this change in the formula for the lower boundary, it is necessary to divide the region into two parts and find the area of each part separately.

- * From (1) with $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x}$, $a = 0$, and $b = 1$, we obtain

$$A_1 = \int_a^b [f(x) - g(x)] dx$$

$$= \int_0^1 \left[\sqrt{x} - (-\sqrt{x}) \right] dx = 2 \int_0^1 \sqrt{x} dx$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= 2 \left[\frac{x^{3/2}}{3/2} \right]_0^1 = \frac{4}{3} \left[(1)^{3/2} - (0)^{3/2} \right]$$

$$= \frac{4}{3} [1 - 0] = \boxed{\frac{4}{3}}.$$

* From (1) with $f(x) = \sqrt{x}$, $g(x) = x - 2$, $a = 1$, and $b = 4$, we obtain

$$\begin{aligned} A_2 &= \int_a^b [f(x) - g(x)] dx \\ &= \int_1^4 [\sqrt{x} - (x - 2)] dx = \int_1^4 (\sqrt{x} - x + 2) dx \\ &= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x \right]_1^4 \\ &= \left[\frac{(4)^{3/2}}{3/2} - \frac{(4)^2}{2} + 2(4) \right] - \left[\frac{(1)^{3/2}}{3/2} - \frac{(1)^2}{2} + 2(1) \right] \\ &= \left(\frac{16}{3} - 8 + 8 \right) - \left(\frac{2}{3} - \frac{1}{2} + 2 \right) = \boxed{\frac{19}{6}}. \end{aligned}$$

* Thus, the area of the entire region is

$$A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \boxed{\frac{9}{2}}.$$

Reversing the Roles of x and y :

* Sometimes it is much easier to find the area of a region by integrating with respect to y rather than x . We will now show how this can be done.

6.1.3 Second Area Problem :

Suppose that w and v are continuous functions of y on an interval $[c, d]$ and that

$$w(y) \geq v(y) \quad \text{for } c \leq y \leq d$$

[This means that the curve $x = w(y)$ lies to the right of the curve $x = v(y)$ and that the two can touch but not cross.]

Find the area A of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above and below by the lines $y = d$ and $y = c$ (Figure 6.1.11).

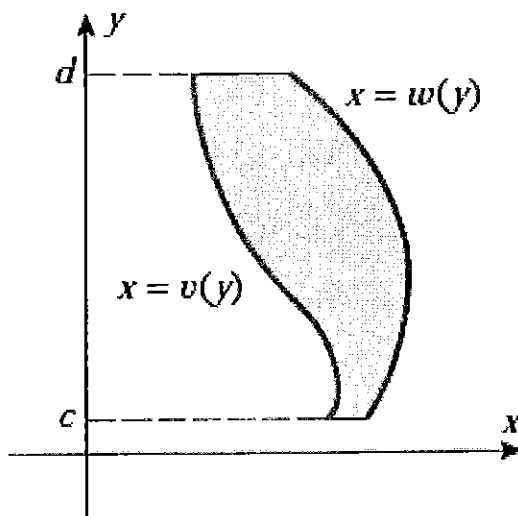


Figure 6.1.11

* Proceeding as in the derivation of (1), but with the roles of x and y reversed, leads to the following analog of 6.1.2.

6.1.4 Area Formula :

If w and v are continuous functions and if $w(x) \geq v(x)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)] dy \quad (4)$$

* The guiding principle in applying this formula is the same as with (1). The integrand in (4) can be viewed as the length of the horizontal cross section at an arbitrary point y on the y -axis, in which case Formula (4) states that the area can be obtained by integrating the length of the horizontal cross section over the interval $[c, d]$ on the y -axis (Figure 6.1.12).

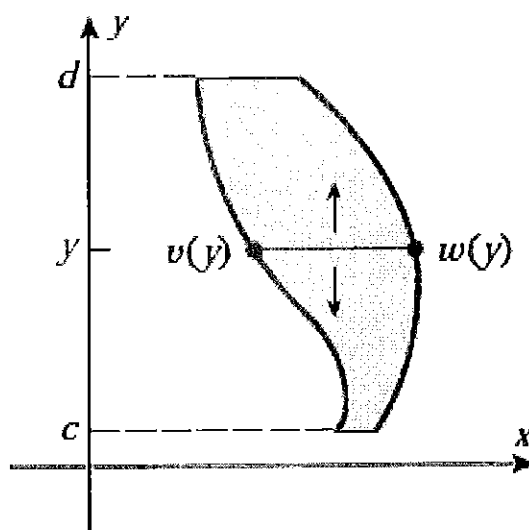


Figure 6.1.12

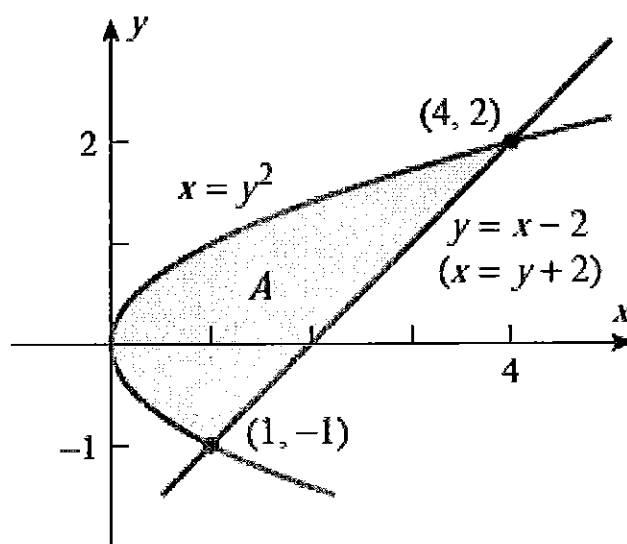
* In Example 4, we split the region into two parts to facilitate integrating with respect to x . In the next example we will

see that splitting this region can be avoided if we integrate with respect to y .

Example 5 :

Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y .

Solution



(a)

Figure 6.1.10

* As indicated in Figure 6.1.10 the left boundary is $x = y^2$, the right boundary is $y = x - 2$, and the region extends over the interval $-1 \leq y \leq 2$. However, to apply (4) the equations for the boundaries must be written so that x is expressed explicitly as a function of y . Thus, we rewrite $y = x - 2$ as $x = y + 2$.

* It now follows from (4) that

$$A = \int_c^d [w(y) - v(y)] dy$$

$$= \int_{-1}^2 \left[(y+2) - (y^2) \right] dy$$

Remember that :

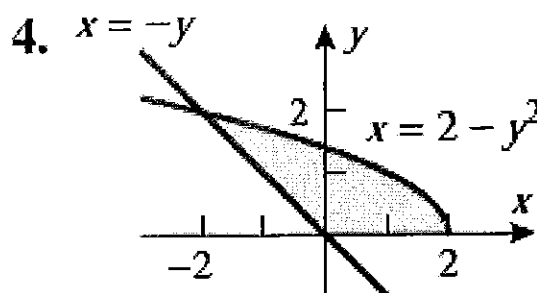
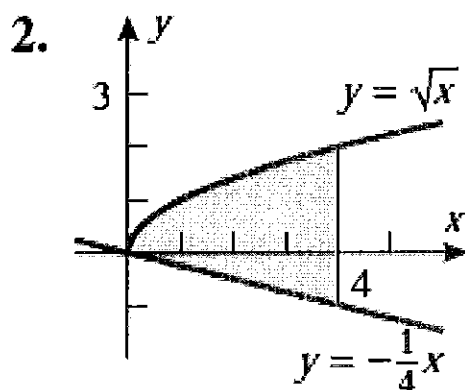
$$* \int y^r dx = \frac{y^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$\begin{aligned} &= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 \\ &= \left[\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3} \right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3} \right] \\ &= \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) = \boxed{\frac{9}{2}}. \end{aligned}$$

which agrees with the result obtained in Example 4.

EXERCISE SET 6.1: (Home Work)

2,4 Find the area of the shaded region.



7,12 Sketch the region enclosed by the curves and find its area.

$$7. \ y = x^2, \ y = \sqrt{x}, \ x = \frac{1}{4}, \ x = 1$$

$$12. \ x^2 = y, \ x = y - 2$$

6.2 VOLUMES BY SLICING; DISKS AND WASHERS:

Volumes by Slicing:

* The idea of finding the volume of a solid is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the volume (Figure 6.2.1).

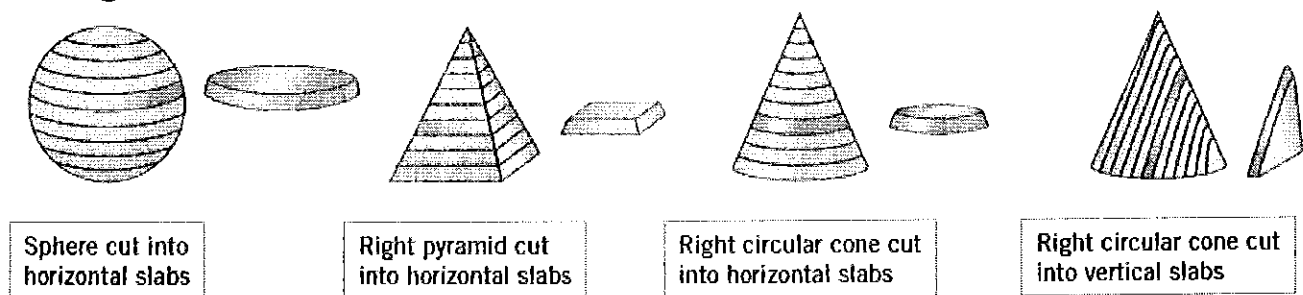


Figure 6.2.1

* If a right cylinder is generated by translating a region of area A through a distance h , then h is called the height (or sometimes the width) of the cylinder, and the volume V of the cylinder is defined to be

$$\boxed{V = A \cdot h = [\text{area of a cross section}] \times [\text{height}]} \quad (2)$$

(Figure 6.2.4).

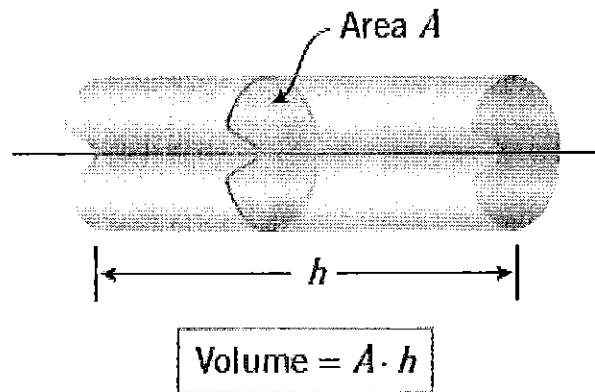


Figure 6.2.4

6.2.1 Problem :

Let S be a solid that extends along the x -axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x -axis at $x = a$ and $x = b$ (Figure 6.2.5). Find the volume V of the solid, assuming that its cross-sectional area $A(x)$ is known at each x in the interval $[a, b]$.

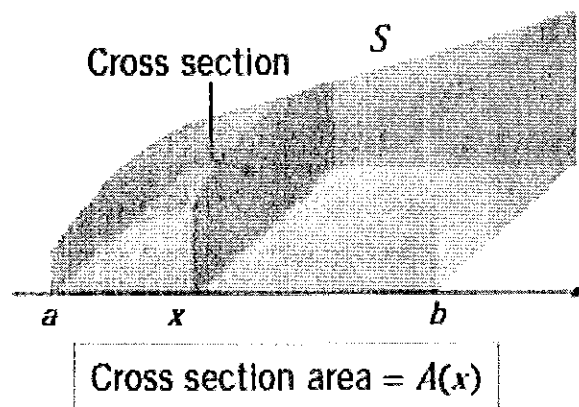


Figure 6.2.5

*** To solve this problem we begin by dividing the interval $[a, b]$ into n subintervals, thereby dividing the solid into n slabs as shown in the left part of Figure 6.2.6.**

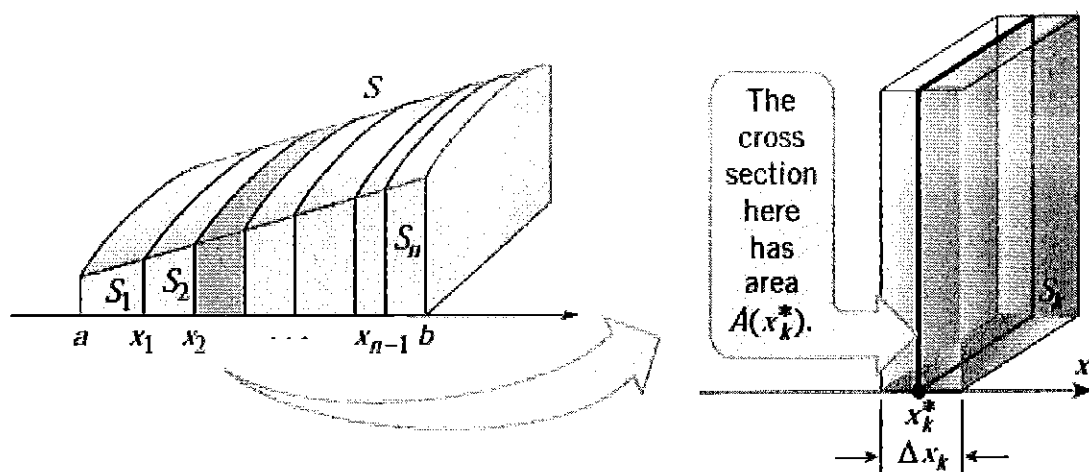


Figure 6.2.6

- * If we assume that the width of the k th subinterval is Δx_k , then the volume of the k th slab can be approximated by the volume $A(x_k^*) \Delta x_k$ of a right cylinder of width (height) Δx_k and cross-sectional area $A(x_k^*)$, where x_k^* is a point in the k th subinterval (see the right of Figure 6.2.6).
- * Adding these approximations yields the following Riemann sum that approximate the volume V :

$$V \approx \sum_{k=1}^n A(x_k^*) \Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n A(x_k^*) \Delta x_k = \int_a^b A(x) dx$$

6.2.2 Volume Formula :

Let S be a solid bounded by two parallel planes perpendicular to the x -axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the

cross-sectional area of S perpendicular to the x -axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x) dx \quad (3)$$

provided $A(x)$ is integrable.

** There is a similar result for cross sections perpendicular to the y -axis.*

6.2.3 Volume Formula :

Page 11

Let S be a solid bounded by two parallel planes perpendicular to the y -axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y -axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y) dy \quad (4)$$

provided $A(y)$ is integrable.

** In words, these formulas (3) and (4) state :*

The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.

Solids of Revolution :

* *A solid of revolution is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region ; the line is called the axis of revolution. Many familiar solids are of this type (Figure 6.2.8).*

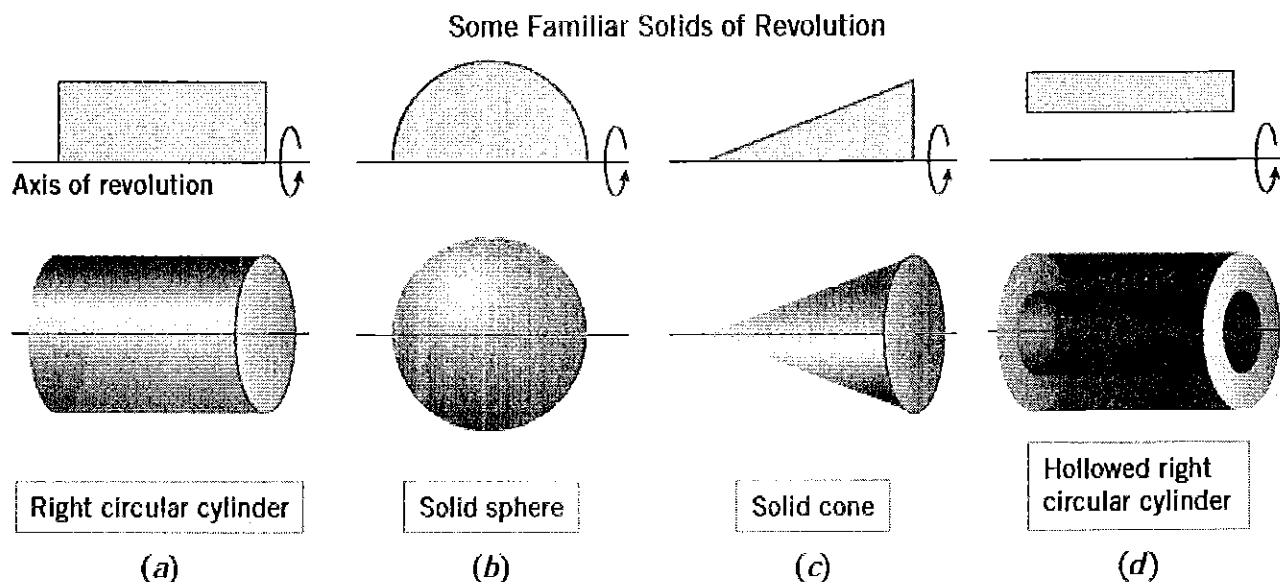


Figure 6.2.8

Volumes by Disks Perpendicular to the x -axis :

6.2.4 Problem :

Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x -axis , and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region R about the x -axis.

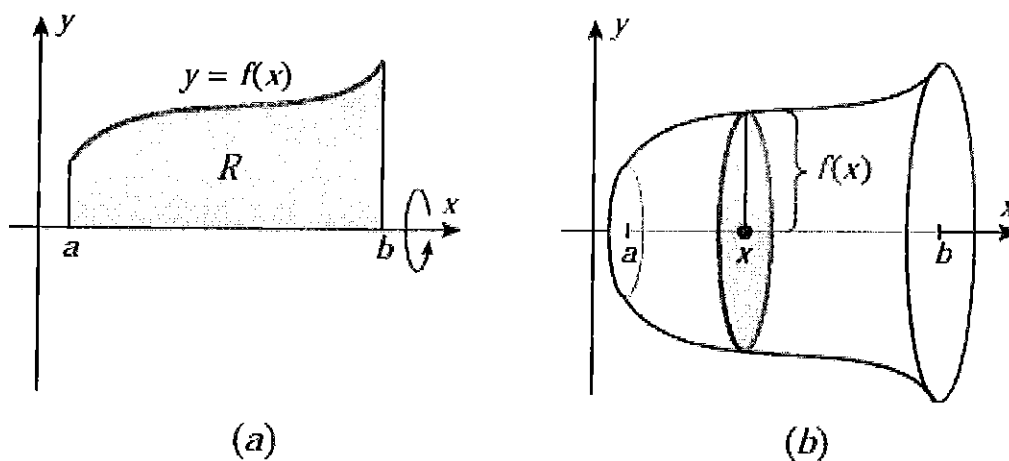


Figure 6.2.9

* We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x -axis at the point x is a circular disk of radius $f(x)$ (Figure 6.2.9b). The area of this region is

$$A(x) = \pi [f(x)]^2$$

Remember that :

* Area of a circle $A = \pi r^2$
 r : radius of circle

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi [f(x)]^2 dx \quad (5)$$

Because the cross sections are disk shaped, the application of this formula is called the method of disks.

Example 2 :

Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis (Figure 6.2.10).

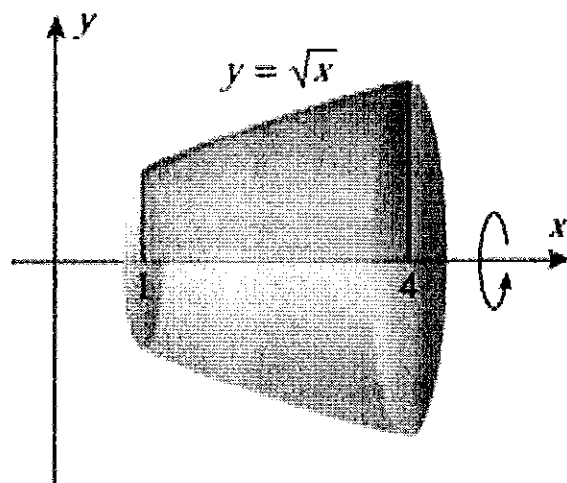


Figure 6.2.10

Solution

** From (5) , the volume is*

$$\begin{aligned} V &= \int_a^b \pi [f(x)]^2 dx \\ &= \int_1^4 \pi [\sqrt{x}]^2 dx = \pi \int_1^4 x dx \end{aligned}$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \pi \left[\frac{x^2}{2} \right]_1^4 = \frac{\pi}{2} [(4)^2 - (1)^2] = \boxed{\frac{15\pi}{2}}.$$

Example 3 :

Derive the formula for the volume of a sphere of radius r .

Solution

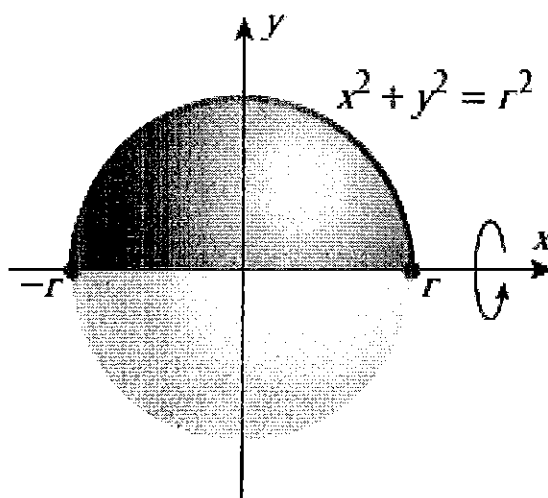


Figure 6.2.11

**** As indicated in Figure 6.2.11 , a sphere of radius r can be generated by revolving the upper semicircular disk enclosed between the x -axis and***

$$x^2 + y^2 = r^2$$

about the x -axis. Since the upper half of this circle is the graph of $y = f(x) = \sqrt{r^2 - x^2}$, it follows from (5) that the volume of the sphere is

$$\begin{aligned} V &= \int_a^b \pi [f(x)]^2 dx \\ &= \int_{-r}^r \pi \left[\sqrt{r^2 - x^2} \right]^2 dx = \pi \int_{-r}^r [r^2 - x^2] dx \end{aligned}$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

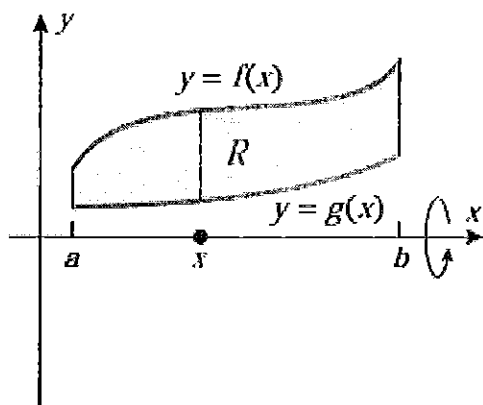
$$\begin{aligned}
&= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r \\
&= \pi \left[\left(r^2 (r) - \frac{(r)^3}{3} \right) - \left(r^2 (-r) - \frac{(-r)^3}{3} \right) \right] \\
&= 2\pi \left(r^3 - \frac{r^3}{3} \right) = 2\pi \left(\frac{2}{3} r^3 \right) = \boxed{\frac{4}{3} \pi r^3}.
\end{aligned}$$

Volumes by Washers Perpendicular to the x-axis :

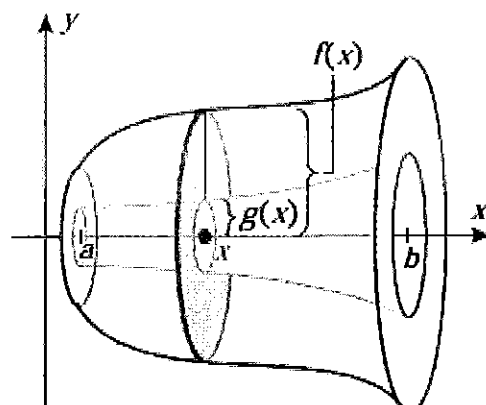
* *Not all solids of revolution have solid interiors ; some have holes or channels that create interior surfaces , as in Figure 6.2.8d. So we will also be interested in problems of the following type.*

6.2.5 Problem :

Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.12a). Find the volume of the solid of revolution that is generated by revolving the region R about the x -axis (Figure 6.2.12b).



(a)



(b)

Figure 6.2.12

* We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x -axis at the point x is the annular or "washer-shaped" region with inner radius $g(x)$ and outer radius $f(x)$ (Figure 6.2.12b); its area is

$$A(x) = \pi [f(x)]^2 - \pi [g(x)]^2 = \pi \left([f(x)]^2 - [g(x)]^2 \right)$$

Thus from (3) the volume of the solid is

$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx \quad (6)$$

* Because the cross sections are washer shaped, the application of this formula is called the method of washers.

Example 4 :

Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x -axis.

Solution

* First sketch the region (Figure 6.2.13a) ; then imagine revolving it about the x-axis (Figure 6.2.13b).

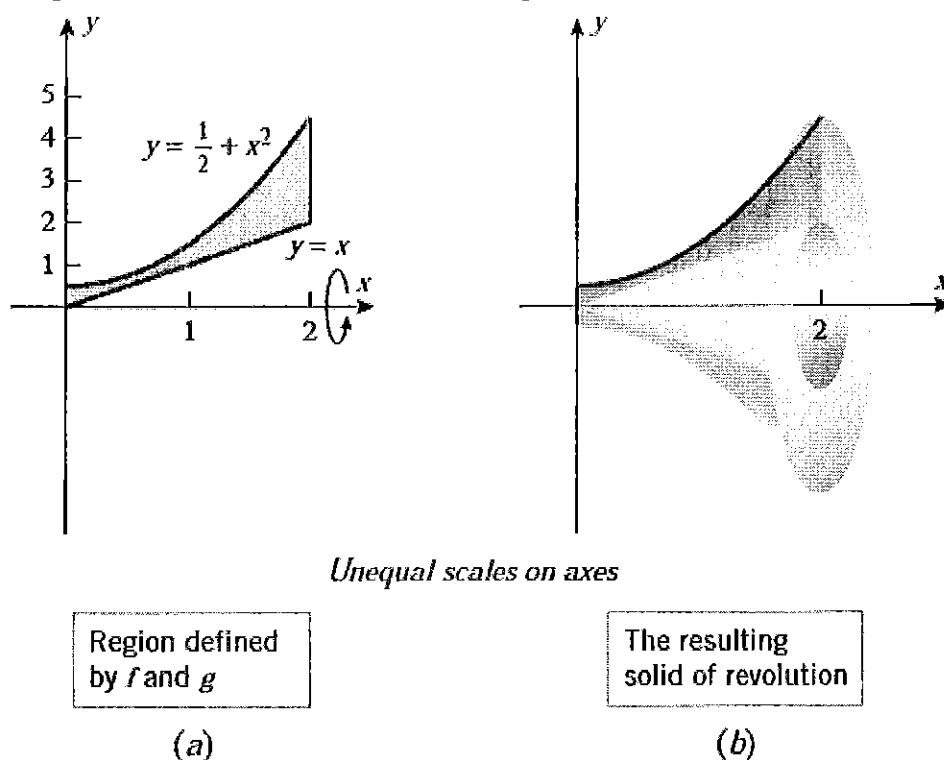


Figure 6.2.13

* From (6) , the volume is

$$\begin{aligned}
 V &= \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx \\
 &= \int_0^2 \pi \left(\left[\frac{1}{2} + x^2 \right]^2 - [x]^2 \right) dx
 \end{aligned}$$

Remember that :

$$* (a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$\begin{aligned}
 &= \pi \int_0^2 \left(\left[\frac{1}{4} + x^2 + x^4 \right] - [x^2] \right) dx \\
 &= \pi \int_0^2 \left(\frac{1}{4} + x^4 \right) dx
 \end{aligned}$$

Remember that :

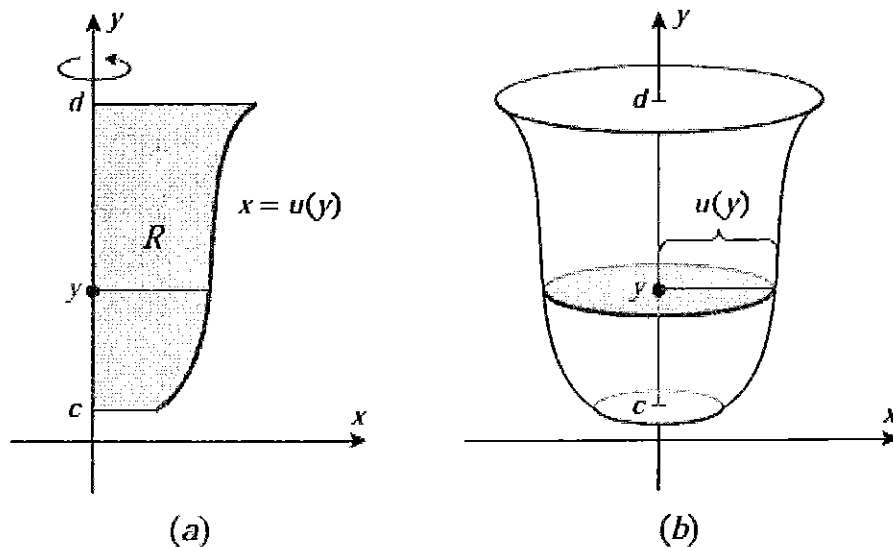
$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \pi \left[\frac{x}{4} + \frac{x^5}{5} \right]_0^2$$

$$= \pi \left[\left(\frac{(2)}{4} + \frac{(2)^5}{5} \right) - \left(\frac{(0)}{4} + \frac{(0)^5}{5} \right) \right] = \boxed{\frac{69\pi}{10}}.$$

Volumes by Disks and Washers Perpendicular to the y-axis :

* The methods of disks and washers have analogs for regions that are revolved about the y-axis (Figures 6.2.14 and 6.2.15). Using the method of slicing and Formula (4) , you should be able to deduce the following formulas for the volumes of the solids in the figures.



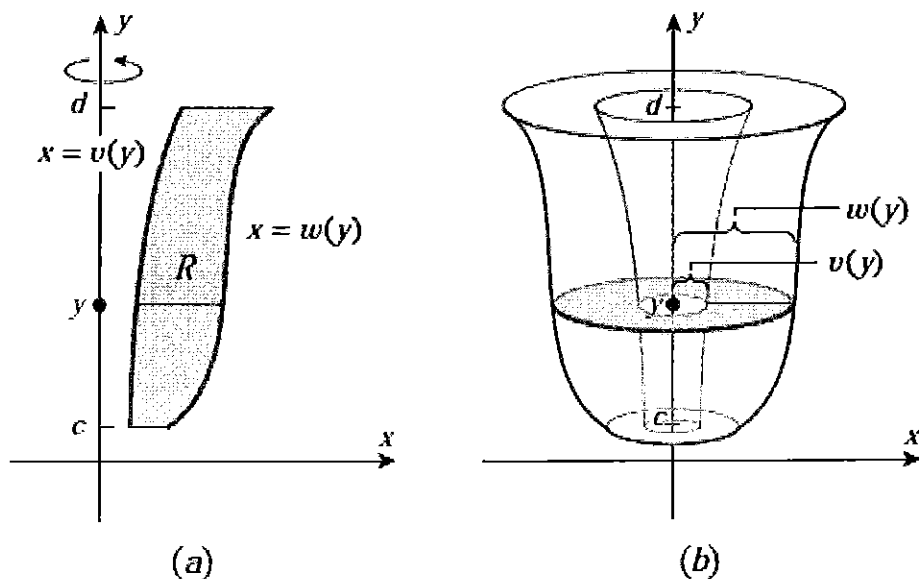
Disks

Figure 6.2.14

$$V = \int_c^d \pi [u(y)]^2 dy$$

Disks

(7)



Washers

Figure 6.2.15

$$V = \int_c^d \pi \left([w(y)]^2 - [v(y)]^2 \right) dy$$

Washers

(8)

Example 5 :

Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y -axis.

Solution

*** First sketch the region and the solid (Figure 6.2.16).**

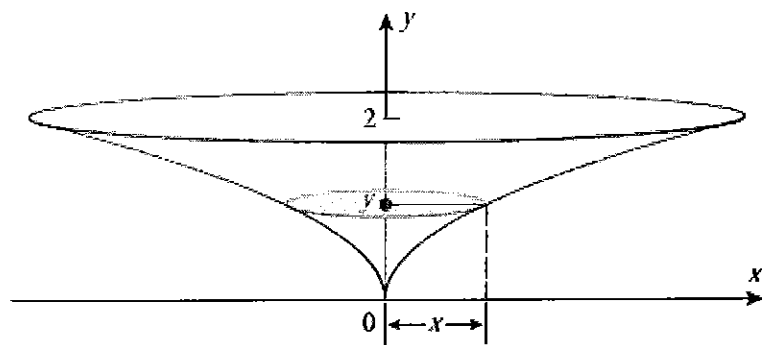
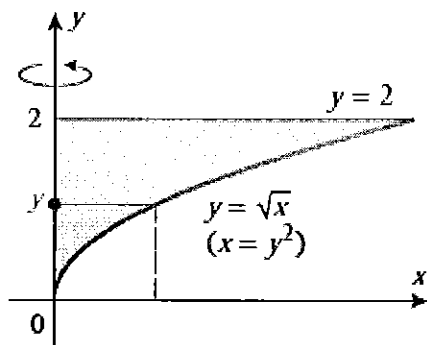


Figure 6.2.16

* The cross section taken perpendicular to the y-axis are disks , so we will apply (7). But first we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus , from (7) with $u(y) = y^2$, the volume is

$$\begin{aligned} V &= \int_c^d \pi [u(y)]^2 dy \\ &= \int_0^2 \pi [y^2]^2 dy = \pi \int_0^2 y^4 dy \end{aligned}$$

Remember that :

$$* \int y^r dy = \frac{y^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$\begin{aligned} &= \pi \left[\frac{y^5}{5} \right]_0^2 \\ &= \frac{\pi}{5} [(2)^5 - (0)^5] = \boxed{\frac{32\pi}{5}}. \end{aligned}$$

Other Axes of Revolution :

* It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. Instead of developing a new formula for each situation, we will appeal to Formulas (3) and (4) and integrate an appropriate cross-sectional area to find the volume.

Example 6 :

Find the volume of the solid generated when the region under the curve $y = x^2$ over the interval $[0, 2]$ is rotated about the line $y = -1$.

Solution

* First sketch the region and the axis of revolution ; then imagine revolving the region about the axis (Figure 6.2.17).

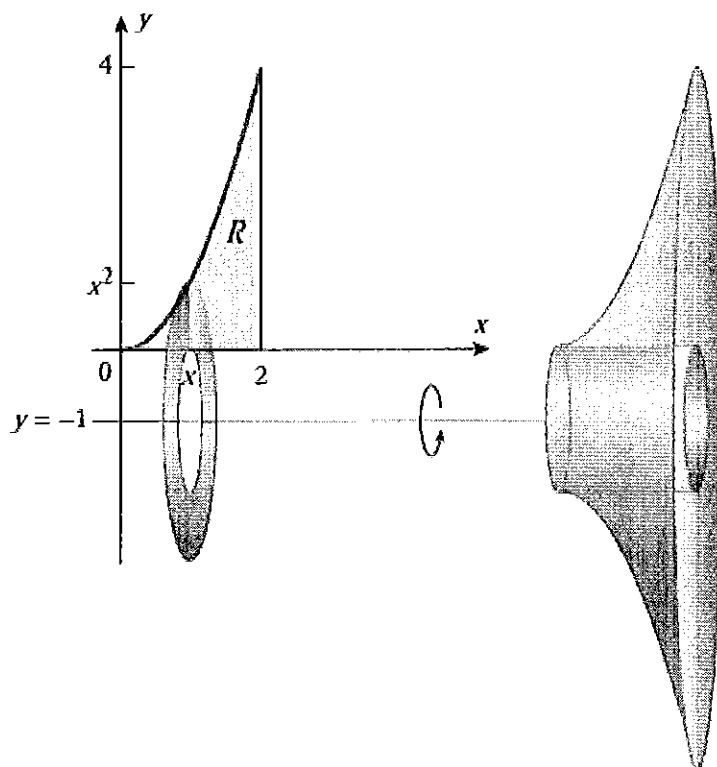


Figure 6.2.17

* At each x in the interval $0 \leq x \leq 2$, the cross section of the solid perpendicular to the axis $y = -1$ is a washer with outer radius $x^2 + 1$ and inner radius 1. Since the area of this washer is

$$A(x) = \pi \left([x^2 + 1]^2 - [1]^2 \right)$$

Remember that :

$$* (a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$\begin{aligned} &= \pi \left([x^2 + 1]^2 - [1]^2 \right) = \pi \left([x^4 + 2x^2 + 1] - [1] \right) \\ &= \pi (x^4 + 2x^2) \end{aligned}$$

it follows by (3) that the volume of the solid is

$$V = \int_a^b A(x) dx = \int_0^2 \pi (x^4 + 2x^2) dx$$

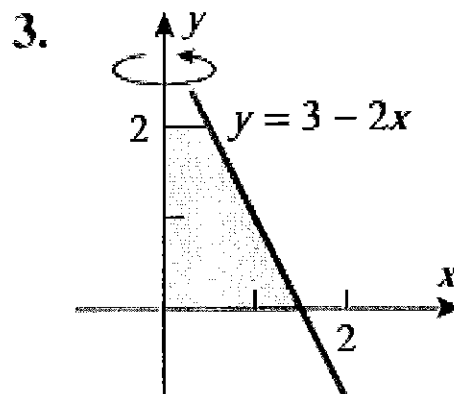
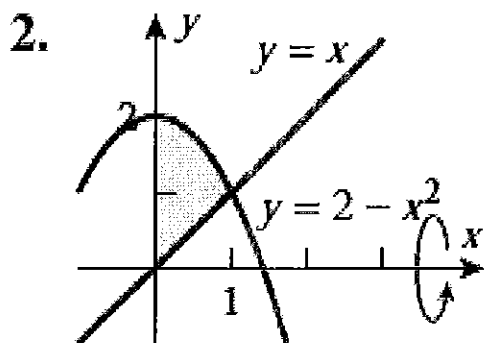
Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$\begin{aligned} &= \pi \left[\frac{x^5}{5} + 2 \frac{x^3}{3} \right]_0^2 \\ &= \pi \left[\left(\frac{(2)^5}{5} + 2 \frac{(2)^3}{3} \right) - \left(\frac{(0)^5}{5} + 2 \frac{(0)^3}{3} \right) \right] = \boxed{\frac{176 \pi}{15}}. \end{aligned}$$

EXERCISE SET 6.2: (Home Work)

2,3 Find the volume of the solid that results when the shaded region is revolved about the indicated axis.



12. Find the volume of the solid that results when the region enclosed by the curves $y = 9 - x^2$, $y = 0$ is revolved about the x-axis.

22. Find the volume of the solid that results when the region enclosed by the curves $y = x^2$, $x = y^2$ is revolved about the y-axis.

6.4 LENGTH OF A PLANE CURVE :

Arc Length :

* Our first objective is to define what we mean by the length (also called the arc length) of a plane curve $y = f(x)$ over an interval $[a, b]$ (Figure 6.4.1). Once that is done we will be able to focus on the problem of computing arc lengths. To avoid some complications that would otherwise occur, we will impose the requirement that f' be continuous on $[a, b]$, in which case we will say that $y = f(x)$ is a smooth curve on $[a, b]$ or that f is a smooth function on $[a, b]$. Thus, we will be concerned with the following problem.

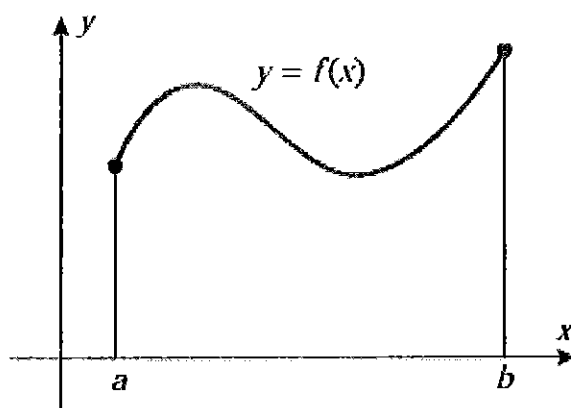


Figure 6.4.1

6.4.1 Arc Length Problem :

Suppose that $y = f(x)$ is a smooth curve on the interval $[a, b]$. Define and find a formula for the arc length L of the curve $y = f(x)$ over the interval $[a, b]$.

* To define the arc length of a curve we start by breaking the curve into small segments. Then we approximate the curve segments by line segments and add the lengths of the line

segments to form a Riemann sum. Figure 6.4.2 illustrates how such line segments tend to become better and better approximations to a curve as the number of segments increases. As the number of segments increases, the corresponding Riemann sums approach a definite integral whose value we will take to be the arc length L of the curve.

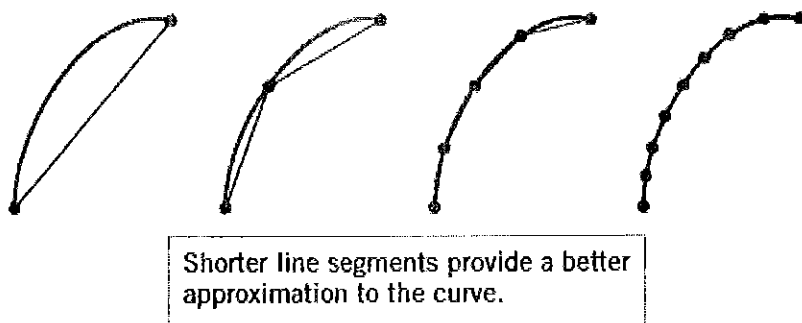


Figure 6.4.2

- * To implement our idea for solving Problem 6.4.1, divide the interval $[a, b]$ into n subintervals by inserting points $x_1, x_2, \dots, x_{n-1}$ between $a = x_0$ and $b = x_n$. As shown in Figure 6.4.3a, let $P_0, P_1, \dots, P_n$ be the points on the curve with x -coordinates $a = x_0, x_1, x_2, \dots, x_{n-1}, b = x_n$, and join these points with straight line segments.

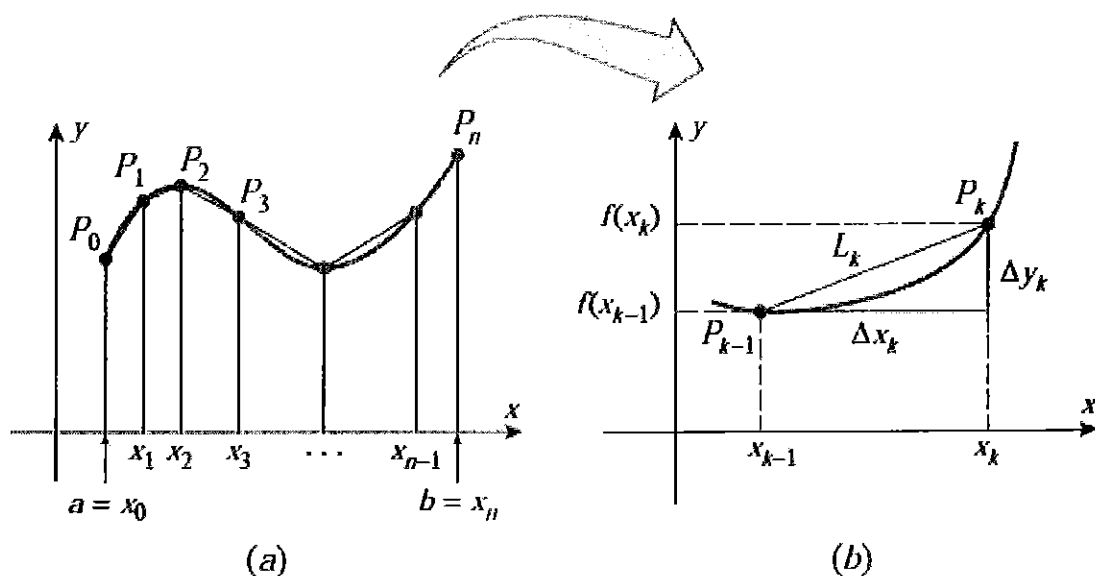


Figure 6.4.3

These line segments form a polygonal path that we can regard as an approximation to the curve $y = f(x)$. As indicated in Figure 6.4.3b, the length L_k of the k th line segment in the polygonal path is

$$\begin{aligned} L_k &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \end{aligned} \quad (1)$$

* If we now add the lengths of these line segments, we obtain the following approximation to the length L of the curve

$$L \approx \sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \quad (2)$$

* To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2) [see revision of Calculus I - MATH 203]. This theorem implies that there is a point x_k^* between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)$$

$$\text{or} \quad f(x_k) - f(x_{k-1}) = f'(x_k^*) \Delta x_k$$

and hence we can rewrite (2) as

$$L \approx \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} = \sum_{k=1}^n \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

* Thus, taking the limit as n increases and the widths of all the subintervals approach zero yields the following integral that defines the arc length L :

$$L = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

* In summary , we have the following definition.

6.4.2 Definition :

If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad (3)$$

* This result provides both a definition and a formula for computing arc lengths. Where convenient , (3) can also be expressed as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (4)$$

* Moreover , for a curve expressed in the form $x = g(y)$, where g' is continuous on $[c, d]$, the arc length L from $y = c$ to $y = d$ can be expressed as

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \quad (5)$$

Example 1 :

Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$ (Figure 6.4.4).

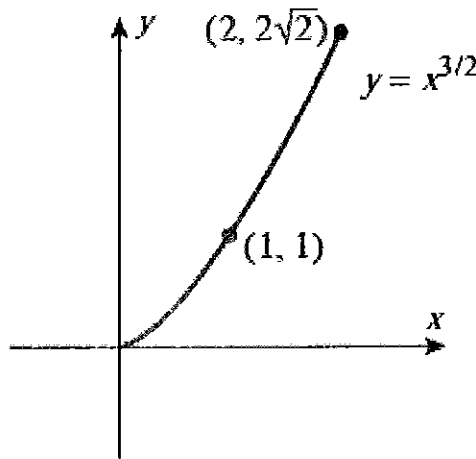


Figure 6.4.4

Solution

$$y = x^{3/2}, \quad \frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

* Since the curve extends from $x = 1$ to $x = 2$, it follows from (4) that

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_1^2 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} dx = \int_1^2 \sqrt{1 + \frac{9}{4}x} dx \end{aligned}$$

* To evaluate this integral we make the u -substitution

$$\boxed{u = 1 + \frac{9}{4}x} \quad \text{so that} \quad du = \frac{9}{4} dx$$

$$\text{or } \frac{4}{9} du = dx$$

With this substitution ,

$$\text{if } x = 1 \quad , \quad u = 1 + \frac{9}{4}(1) = \frac{13}{4}$$

$$\text{if } x = 2 \quad , \quad u = 1 + \frac{9}{4}(2) = \frac{22}{4}$$

$$L = \frac{4}{9} \int_{13/4}^{22/4} \sqrt{u} \, du$$

Remember that :

$$* \int u^r \, du = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$\begin{aligned} &= \frac{4}{9} \left[\frac{u^{3/2}}{3/2} \right]_{13/4}^{22/4} \\ &= \frac{8}{27} \left[\left(\frac{22}{4} \right)^{3/2} - \left(\frac{13}{4} \right)^{3/2} \right] \\ &= \frac{8}{27} \left[\frac{22\sqrt{22}}{4\sqrt{4}} - \frac{13\sqrt{13}}{4\sqrt{4}} \right] \\ &= \boxed{\frac{22\sqrt{22} - 13\sqrt{13}}{27} \approx 2.09} . \end{aligned}$$

EXERCISE SET 6.4: (Home Work)

3,5 Find the exact arc length of the curve over the interval.

3. $y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$

5. $y = x^{2/3}$ from $x = 1$ to $x = 8$

CHAPTER (6)

PRINCIPLES OF INTEGRAL EVALUATION

7.1 AN OVERVIEW OF INTEGRATION METHODS :

Methods for Approaching Integration Problems :

- * *There are three basic approaches for evaluating unfamiliar integrals :*
 - *Technology – CAS programs such as Mathematica , Maple , and the open source program Sage are capable of evaluating extremely complicated integrals , and such programs are increasingly available for both computers and handheld calculators.*
 - *Tables – Prior to development of CAS programs , scientist relied heavily on tables to evaluate difficult integrals arising in applications. More comprehensive tables appear in various reference books such as the CRC Standard Mathematical Tables and Formulas , CRC press , Inc. , 2002.*
 - *Transformation Methods – Transformation methods are methods for converting unfamiliar integrals into familiar integrals. These include u-substitution , algebraic manipulation of the integrand , and other methods we will discuss in this chapter.*

A Review of Familiar Integration Formulas :

* The following is a list of basic integrals.

Constants , Power , Exponentials :

$$1. \int du = u + C$$

$$2. \int a du = a \int du = au + C$$

$$3. \int u^r du = \frac{u^{r+1}}{r+1} + C, \quad r \neq -1$$

$$4. \int \frac{du}{u} = \ln|u| + C$$

$$5. \int e^u du = e^u + C$$

$$6. \int b^u du = \frac{b^u}{\ln b} + C, \quad b > 0, b \neq 1$$

Trigonometric Functions :

$$7. \int \sin u du = -\cos u + C$$

$$8. \int \cos u du = \sin u + C$$

$$9. \int \sec^2 u du = \tan u + C$$

$$10. \int \csc^2 u du = -\cot u + C$$

$$11. \int \sec u \tan u du = \sec u + C$$

$$12. \int \csc u \cot u du = -\csc u + C$$

$$13. \int \tan u dx = -\ln|\cos u| + C$$

$$14. \int \cot u dx = \ln|\sin u| + C$$

Hyperbolic Functions :

$$15. \int \sinh u dx = \cosh u + C$$

$$16. \int \cosh u du = \sinh u + C$$

$$17. \int \sec h^2 u du = \tanh u + C$$

$$18. \int \csc h^2 u du = -\coth u + C$$

$$19. \int \sec u \tan u du = -\sec hu + C$$

$$20. \int \csc hu \coth u du = -\csc hu + C$$

Algebraic Functions $(a > 0)$ _:

$$21. \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C \quad (|u| < a)$$

$$22. \int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1} \left(\frac{u}{a} \right) + C = \ln \left(u + \sqrt{a^2 + u^2} \right) + C$$

$$23. \int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1} \left(\frac{u}{a} \right) + C = \ln \left| u + \sqrt{u^2 - a^2} \right| + C \quad (0 < a < |u|)$$

$$24. \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$$

$$25. \int \frac{du}{a^2 - u^2} = \frac{1}{a} \tanh^{-1} \left(\frac{u}{a} \right) + C = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C \quad (|u| < a)$$

$$26. \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + C \quad (0 < a < |u|)$$

$$27. \int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \operatorname{sech}^{-1} \left| \frac{u}{a} \right| + C$$

$$= -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C \quad (0 < |u| < a)$$

$$28. \int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \operatorname{csc} h^{-1} \left| \frac{u}{a} \right| + C = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 + u^2}}{u} \right| + C$$

EXERCISE SET 7.1:

(Home Work)

Evaluate the integrals by making appropriate u -substitution and applying the formulas reviewed in this section.

$$1. \int (4 - 2x)^3 dx$$

$$3. \int x \sec^2(x^2) dx$$

$$5. \int \frac{\sin 3x}{2 + \cos 3x} dx$$

$$6. \int \frac{1}{9 + 4x^2} dx$$

$$7. \int e^x \sin(e^x) dx$$

$$9. \int e^{\tan x} \sec^2 x dx$$

$$11. \int \cos^5 5x \sin 5x dx$$

$$13. \int \frac{e^x}{\sqrt{4 - e^{2x}}} dx$$

$$14. \int \frac{e^{\tan^{-1} x}}{1 + x^2} dx$$

$$24. \int \frac{\cos(\ln x)}{x} dx$$

7.2 INTEGRATION BY PARTS :

The Product Rule and Integration by Parts :

* Our primary goal in this section is to develop a general method for attacking integrals of the form

$$\int f(x) g(x) dx$$

* As a first step, let $G(x)$ be any antiderivative of $g(x)$. In this case $G'(x) = g(x)$, so the product rule for differentiating $f(x) G(x)$ can be expressed as

$$\begin{aligned} \frac{d}{dx}[f(x) G(x)] &= f(x) G'(x) + f'(x) G(x) \\ &= f(x) g(x) + f'(x) G(x) \end{aligned} \quad (1)$$

* This implies that $f(x) G(x)$ is an antiderivative of the function on the right side of (1), so we can express (1) in integral form as

$$\int [f(x) g(x) + f'(x) G(x)] dx = f(x) G(x)$$

or, equivalently, as

$$\boxed{\int f(x) g(x) dx = f(x) G(x) - \int f'(x) G(x) dx} \quad (2)$$

* This formula allows us to integrate $f(x) g(x)$ by integrating $f'(x) G(x)$ instead, and in many cases the net effect is to replace a difficult integration with an easier one. The application of this formula is called integration by parts.

* In practice, we usually rewrite (2) by letting

$$u = f(x) \quad , \quad du = f'(x) dx$$

$$v = G(x) \quad , \quad dv = G'(x) dx = g(x) dx$$

This yields the following alternative form for (2) :

$$\boxed{\int u dv = uv - \int v du} \quad (3)$$

Example 1 :

Use integration by parts to evaluate $\int x \cos x dx$.

Solution

* We will apply Formula (3). The first step is to make a choice for u and dv to put the given integral in the form $\int u dv$. We will let

$$u = x \quad \text{and} \quad dv = \cos x dx$$

(Other possibilities will be considered latter).

* The second step is to compute du from u and v from dv . This yields

$$du = dx \quad \text{and} \quad v = \int dv = \int \cos x dx = \sin x$$

Remember that :

$$* \int \cos x dx = \sin x + C$$

* The third step is to apply Formula (3). This yields

Remember that :

$$* \int u dv = uv - \int v du$$

$$\begin{aligned} \int \underbrace{x}_{u} \underbrace{\cos x dx}_{dv} &= \underbrace{x}_{u} \underbrace{\sin x}_v - \int \underbrace{\sin x}_v \underbrace{dx}_{du} \\ &= x \sin x - (-\cos x) + C \\ &= \boxed{x \sin x + \cos x + C}. \end{aligned}$$

Note :

- * In Example 1 we omitted the constant of integration in calculating v from dv .

Guidelines for Integration by Parts :

- * The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original. In general, there are no hard and fast rules for doing this; it is mainly a matter of experience that comes from lots of practice. A strategy that often works is to choose u and dv so that u becomes "simpler" when differentiated, while leaving a dv that can be readily integrated to obtain v . Thus, for the integral $\int x \cos x \, dx$ in Example 1, both goals were achieved by letting $u = x$ and $dv = \cos x \, dx$. In contrast, $u = \cos x$ would not have been a good first choice in that example, since $du/dx = -\sin x$ is no simpler than u . Indeed, had we chosen

$$\begin{array}{ll} u = \cos x & dv = x \, dx \\ du = -\sin x \, dx & v = \int x \, dx = \frac{x^2}{2} \end{array}$$

Remember that :

$$* \int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

then we would have obtained

Remember that :

$$* \int u \, dv = uv - \int v \, du$$

$$\int \underbrace{\cos x}_u \underbrace{x dx}_{dv} = \underbrace{\cos x}_u \cdot \underbrace{\frac{x^2}{2}}_v - \int \underbrace{\frac{x^2}{2}}_v \underbrace{(-\sin x) dx}_{du}$$

$$= \frac{x^2}{2} \cos x + \frac{1}{2} \int x^2 \sin x dx$$

For this choice of u and dv , the new integral is actually more complicated than the original.

- * There is another useful strategy for choosing u and dv that can be applied when the integral is a product of two functions from different categories in the list*

*Logarithmic, Inverse trigonometric (or Inverse hyperbolic),
Algebraic, Trigonometric (or Hyperbolic), Exponential*

In this case you will often be successful if you take u to be the function whose category occurs earlier in the list and take dv to be the rest of the integrand. The acronym LIATE will help you to remember the order. The method does not work all the time, but it works often enough to be useful.

- * Note, for example, that the integrand in Example 1 consists of the product of the algebraic function x and the trigonometric function $\cos x$. Thus, the LIATE method suggests that we should let $u = x$ and $dv = \cos x dx$, which proved to be a successful choice.*

Example 2 :

Evaluate $\int x e^x dx$.

Solution

* In this case the integrand is the product of the algebraic function x with the exponential function e^x . According to LIATE we should let

$$u = x \qquad dv = e^x dx$$

$$du = dx \qquad v = \int e^x dx = e^x$$

Remember that :

$$* \int e^x dx = e^x + C$$

Thus , from (3)

Remember that :

$$* \int u dv = uv - \int v du$$

$$\begin{aligned} \int \underset{u}{x} \underset{dv}{e^x} dx &= \underset{u}{x} \underset{v}{e^x} - \int \underset{v}{e^x} \underset{du}{dx} \\ &= \boxed{x e^x - e^x + C} . \end{aligned}$$

Example 3 :

Evaluate $\int \ln x dx$.

Solution

* In this case the integrand is the product of the logarithmic function $\ln x$ with the algebraic function 1 . According to LIATE we should let

$$u = \ln x \qquad dv = 1 dx$$

$$du = \frac{1}{x} dx \qquad v = \int 1 dx = x$$

Remember that :

$$* \int 1 \, dx = x + C$$

Thus , from (3)

Remember that :

$$* \int u \, dv = uv - \int v \, du$$

$$\begin{aligned} \int \ln x \, 1 \, dx &= \ln x \cdot x - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - \int dx \\ &= \boxed{x \ln x - x + C} . \end{aligned}$$

Repeated Integration by Parts :

* It is sometimes necessary to use integration by parts more than once in the same problem.

Example 4 :

Evaluate $\int x^2 e^{-x} \, dx$.

Solution

* In this case the integrand is the product of the algebraic function x^2 with the exponential function e^{-x} . According to LIATE we should let

$$u = x^2 \qquad dv = e^{-x} \, dx$$

$$du = 2x \, dx \qquad v = \int e^{-x} \, dx = -e^{-x}$$

Remember that :

$$* \int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$$

Thus , from (3)

Remember that :

$$* \int u dv = uv - \int v du$$

$$\begin{aligned} \int \underbrace{x^2}_u \underbrace{e^{-x} dx}_{dv} &= \underbrace{x^2}_u \underbrace{(-e^{-x})}_v - \int \underbrace{(-e^{-x})}_v \underbrace{2x dx}_{du} \\ &= -x^2 e^{-x} + 2 \int x e^{-x} dx \end{aligned} \quad (4)$$

* The last integral is similar to the original except that we replaced x^2 by x . Another integration by parts applied to will complete the problem. We let

$$\begin{aligned} u &= x & dv &= e^{-x} dx \\ du &= dx & v &= \int e^{-x} dx = -e^{-x} \end{aligned}$$

Thus ,

$$\begin{aligned} \int \underbrace{x}_u \underbrace{e^{-x} dx}_{dv} &= \underbrace{x}_u \underbrace{(-e^{-x})}_v - \int \underbrace{(-e^{-x})}_v \underbrace{dx}_{du} \\ &= -x e^{-x} + \int e^{-x} dx \\ &= -x e^{-x} - e^{-x} + C \end{aligned}$$

* Finally , substituting this into the last line of (4) yields

$$\begin{aligned} \int x^2 e^{-x} dx &= -x^2 e^{-x} + 2 \int x e^{-x} dx \\ &= -x^2 e^{-x} + 2(-x e^{-x} - e^{-x}) + C \\ &= \boxed{-(x^2 + 2x + 2)e^{-x} + C} . \end{aligned}$$

* The LIATE method suggests that integrals of the form

$$\int e^{ax} \sin bx \, dx \quad \text{and} \quad \int e^{ax} \cos bx \, dx$$

can be evaluated by letting $u = \sin bx$ or $u = \cos bx$ and $dv = e^{ax} \, dx$. However, this will require a technique that deserves special attention.

Example 5 :

Evaluate $\int e^x \cos x \, dx$.

Solution

* In this case the integrand is the product of the trigonometric function $\cos x$ with the exponential function e^x . According to LIATE we should let

$$\begin{aligned} u &= \cos x & dv &= e^x \, dx \\ du &= -\sin x \, dx & v &= \int e^x \, dx = e^x \end{aligned}$$

Remember that :

$$* \int e^x \, dx = e^x + C$$

Thus, from (3)

Remember that :

$$* \int u \, dv = uv - \int v \, du$$

$$\begin{aligned} \int \underbrace{\cos x}_u \underbrace{e^x \, dx}_{dv} &= \underbrace{\cos x}_u \cdot \underbrace{e^x}_v - \int \underbrace{e^x}_v \underbrace{(-\sin x \, dx)}_{du} \\ &= e^x \cos x + \int e^x \sin x \, dx \end{aligned} \quad (5)$$

* Since the integral $\int e^x \sin x \, dx$ is similar in form to the original integral $\int e^x \cos x \, dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$\begin{aligned} u &= \sin x & dv &= e^x dx \\ du &= \cos x dx & v &= \int e^x dx = e^x \end{aligned}$$

Thus ,

$$\int \underbrace{\sin x}_u \underbrace{e^x}_{dv} dx = \underbrace{\sin x}_u \cdot \underbrace{e^x}_v - \int \underbrace{e^x}_v \underbrace{\cos x dx}_{du}$$

* Together with Equation (5) this yields

$$\int e^x \cos x dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx$$

we obtain

$$2 \int e^x \cos x dx = e^x \cos x + e^x \sin x$$

and hence

$$\int e^x \cos x dx = \boxed{\frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C}.$$

* Note that the integral $\int e^x \cos x dx$ can be evaluated by letting $u = e^x$ and $dv = \cos x dx$.

A Tabular Method for Repeated Integration by Parts :

* Integrals of the form

$$\int p(x) f(x) dx$$

where $p(x)$ is a polynomial , can sometimes be evaluated using repeated integration by parts in which u is taken to be $p(x)$ or one of its derivatives at each stage. Since du is computed by differentiating u , the repeated differentiation of $p(x)$ will eventually produce 0 , at which point you may be left with a simplified integration problem. A convenient

method for organizing the computations into two columns is called *tabular integration by parts*.

Tabular Intagration by Parts :

- Step 1.** Differentiate $p(x)$ repeatedly until you obtain 0 , and list the results in the first column.
- Step 2.** Integrate $f(x)$ repeatedly and list the result in the second column.
- Step 3.** Draw an arrow from each entry in the first column to the entry that is one row down in the second column.
- Step 4.** Label the arrows with alternating + and - signs , starting with a + .
- Step 5.** For each arrow , from the product of the expressions at its tip and tail and then multiply that product by +1 or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

* This process is illustrated in Figure 7.2.1 for the integral $\int (x^2 - x) \cos x \, dx$.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
$x^2 - x$	+	$\cos x$
$2x - 1$	-	$\sin x$
2	+	$-\cos x$
0	-	$-\sin x$

Figure 7.2.1

$$\begin{aligned}\int (x^2 - x) \cos x \, dx &= (x^2 - x) \sin x - (2x - 1)(-\cos x) \\ &\quad + 2(-\sin x) + C \\ &= \boxed{(x^2 - x - 2) \sin x + (2x - 1) \cos x + C}.\end{aligned}$$

Example 6 :

Evaluate $\int x^2 \sqrt{x-1} \, dx$ using tabular integration by parts.

Solution

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
x^2	$\searrow +$	$(x-1)^{1/2}$
$2x$	$\searrow -$	$\frac{2}{3}(x-1)^{3/2}$
2	$\searrow +$	$\frac{4}{15}(x-1)^{5/2}$
0	$\searrow$	$\frac{8}{105}(x-1)^{7/2}$

Remember that :

$$* \int (ax+b)^r \, dx = \frac{1}{a} \frac{(ax+b)^{r+1}}{r+1} + C, \quad r \neq -1$$

* Thus, it follows that

$$\begin{aligned}\int x^2 \sqrt{x-1} \, dx \\ = \boxed{\frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C}\end{aligned}$$

* Note that the integral $\int x^2 \sqrt{x-1} \, dx$ can be evaluated by using u-substitution $\boxed{u = x-1}$ or $\boxed{u = \sqrt{x-1}}$.

Integration by Parts for Definite Integrals :

* For definite integrals the formula corresponding to (3) is

$$\boxed{\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du} \quad (7)$$

Remark :

* It is important to keep in mind that the variables u and v in this formula are functions of x and that the limits of integration in (7) are limits on the variable x . Sometimes it is helpful to emphasize this by writing (7) as

$$\boxed{\int_{x=a}^b u \, dv = [uv]_{x=a}^b - \int_{x=a}^b v \, du} \quad (8)$$

* The next example illustrates how integration by parts can be used to integrate the inverse trigonometric functions.

Example 7 :

Evaluate $\int_0^1 \tan^{-1} x \, dx$.

Solution

* In this case the integrand is the product of the inverse trigonometric function $\tan^{-1} x$ with the algebraic function 1. According to LIATE we should let

$$\begin{aligned} u &= \tan^{-1} x & dv &= 1 \, dx \\ du &= \frac{1}{1+x^2} \, dx & v &= \int 1 \, dx = x \end{aligned}$$

Remember that :

$$* \int 1 \, dx = x + C$$

Thus , from (7)

Remember that :

$$* \int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du$$

$$\begin{aligned} \int_0^1 \underbrace{\tan^{-1} x}_u \underbrace{1 \, dx}_{dv} &= \left[\underbrace{\tan^{-1} x}_u \cdot \underbrace{(x)}_v \right]_0^1 - \int_0^1 \underbrace{x}_v \underbrace{\frac{1}{1+x^2} \, dx}_{du} \\ &= \left[x \tan^{-1} x \right]_0^1 - \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} \, dx \end{aligned}$$

Remember that :

*

$$\int [g(x)]^r g'(x) \, dx = \frac{[g(x)]^{r+1}}{r+1} + C, \, r \neq -1$$

$$* \int \frac{g'(x)}{g(x)} \, dx = \ln |g(x)| + C$$

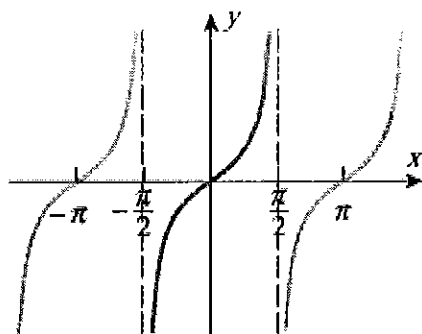
$$* \int \frac{g'(x)}{\sqrt{g(x)}} \, dx = 2\sqrt{g(x)} + C$$

$$\begin{aligned} &= \left[x \tan^{-1} x \right]_0^1 - \frac{1}{2} \left[\ln |1+x^2| \right]_0^1 \\ &= \left[(1 \tan^{-1} 1) - (0 \tan^{-1} 0) \right] - \frac{1}{2} \left[(\ln |1+1^2|) - (\ln |1+0^2|) \right] \end{aligned}$$

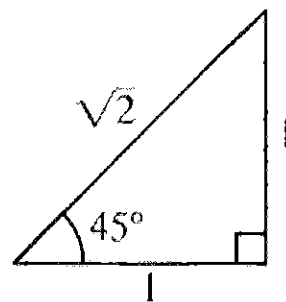
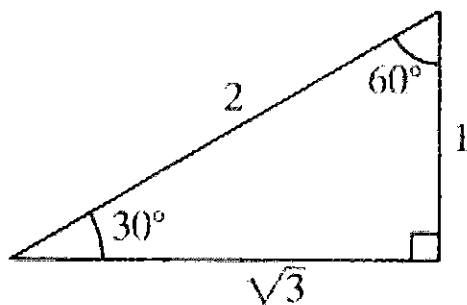
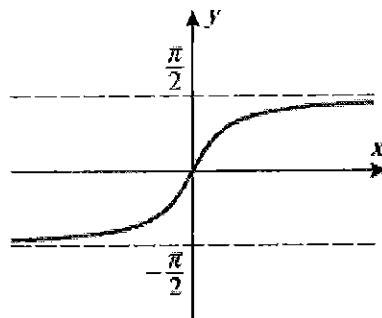
Remember that :

$$* \tan^{-1} 0 = 0, \, \tan^{-1} 1 = \frac{\pi}{4}$$

$$y = \tan x$$



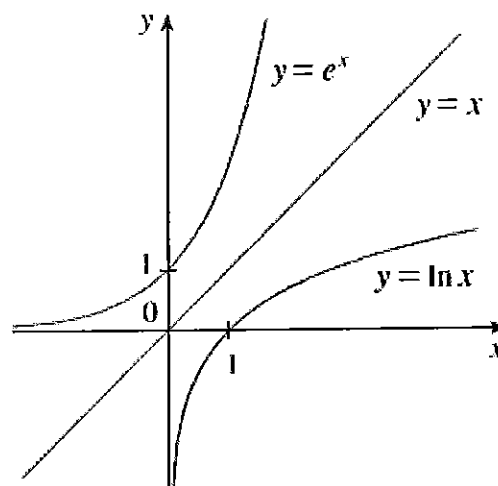
$$y = \tan^{-1} x$$



$$= \left[\frac{\pi}{4} - 0 \right] - \frac{1}{2} [\ln |2| - \ln |1|]$$

Remember that :

$$* \ln 1 = 0, \ln x^r = r \ln x, r \in \mathbb{R}$$



$$= \left[\frac{\pi}{4} \right] - \left[\ln(2)^{1/2} - 0 \right] = \boxed{\frac{\pi}{4} - \ln \sqrt{2} + C}.$$

Reduction Formulas :

* *Integration by parts can be used to derive reduction formulas for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a lower power of that function. For example, if n is a positive integer and $n \geq 2$, then integration by parts can be used to obtain the reduction formulas*

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \quad (9)$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \quad (10)$$

* *To illustrate how such formulas can be obtained, let us derive (10). We begin by writing $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$ and letting*

$$\begin{aligned} u &= \cos^{n-1} x & dv &= \cos x \, dx \\ du &= (n-1) \cos^{n-2} x (-\sin x) \, dx & v &= \int \cos x \, dx = \sin x \\ &= -(n-1) \cos^{n-2} x \sin x \, dx \end{aligned}$$

Remember that :

$$* \int \cos x \, dx = \sin x + C$$

Thus, from (3)

Remember that :

$$* \int u \, dv = uv - \int v \, du$$

$$\int \cos^n x \, dx = \int \underbrace{\cos^{n-1} x}_u \underbrace{\cos x \, dx}_{dv}$$

$$= \underbrace{\cos^{n-1} x}_u \underbrace{\sin x}_v - \int \underbrace{\sin x}_v \underbrace{\left(- (n-1) \cos^{n-2} x \sin x dx \right)}_{du}$$

$$= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x dx$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1$$

$$= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x dx$$

$$\int \cos^n x dx + (n-1) \int \cos^n x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

$$n \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

$$\boxed{\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx .}$$

from which (10) follows.

* The derivation of reduction formula (9) is similar.

We begin by writing $\sin^n x$ as $\sin^{n-1} x \cdot \sin x$ and letting

$$u = \sin^{n-1} x \quad dv = \sin x dx$$

$$du = (n-1) \sin^{n-2} x (\cos x) dx \quad v = \int \sin x dx = -\cos x$$

$$= (n-1) \sin^{n-2} x \cos x dx$$

Remember that :

$$* \int \sin x dx = -\cos x + C$$

Thus , from (3)

Remember that :

$$* \int u dv = uv - \int v du$$

$$\begin{aligned}
 \int \sin^n x \, dx &= \int \underbrace{\sin^{n-1} x}_u \underbrace{\sin x \, dx}_{dv} \\
 &= \underbrace{\sin^{n-1} x}_u \underbrace{(-\cos x)}_v - \int \underbrace{(-\cos x)}_v \underbrace{(n-1) \sin^{n-2} x \cos x \, dx}_{du} \\
 &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx
 \end{aligned}$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$$

$$\int \sin^n x \, dx + (n-1) \int \sin^n x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

$$n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

$$\boxed{\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx .}$$

* *Reduction formula (9) and (10) reduce the exponent of sine (or cosine) by 2. Thus, if the formulas are applied repeatedly, the exponent can eventually be reduced to 0 if n is even or 1 if n is odd, at which point the integration can be completed.*

Example 8 :

Evaluate $\int \cos^4 x \, dx$ using the reduction formula (10)

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx .$$

Solution

* *From reduction formula (10)*

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

with $n = 4$

$$\int \cos^4 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx$$

* Now apply reduction formula with $n = 2$

$$\begin{aligned} \int \cos^4 x \, dx &= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left(\frac{1}{2} \cos x \sin x + \frac{1}{2} \int dx \right) \\ &= \boxed{\frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C} . \end{aligned}$$

EXERCISE SET 7.2:

(Home Work)

Evaluate the integral.

1. $\int x e^{-2x} \, dx$

7. $\int x^2 \cos x \, dx$

9. $\int x \ln x \, dx$

15. $\int \sin^{-1} x \, dx$

19. $\int e^x \sin x \, dx$

27. $\int_0^2 x e^{2x} \, dx$

61.(a) Use reduction formula (9)

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

to evaluate $\int \sin^4 x \, dx$.

7.3 INTEGRATING TRIGONOMETRIC FUNCTIONS :

Integrating Powers of Sine and Cosine :

* We begin by recalling two reduction formulas from the preceding section.

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \quad (1)$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \quad (2)$$

* In the case where $n = 2$, these formula yields

$$\begin{aligned} \int \sin^2 x \, dx &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx \\ &= \frac{1}{2} x - \frac{1}{2} \sin x \cos x + C \end{aligned} \quad (3)$$

$$\begin{aligned} \int \cos^2 x \, dx &= \frac{1}{2} \cos x \sin x + \frac{1}{2} \int dx \\ &= \frac{1}{2} x + \frac{1}{2} \sin x \cos x + C \end{aligned} \quad (4)$$

* Alternative forms of these integration formulas can be derived from the trigonometric identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad (5)$$

and $\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad (6)$

which follow from the double-angle formulas

$$\cos 2x = 1 - 2 \sin^2 x \quad \text{and} \quad \cos 2x = 2 \cos^2 x - 1$$

* *These identities yield*

$$\int \sin^2 x \, dx = \frac{1}{2} \int (1 - \cos 2x) \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \quad (7)$$

Remember that :

$$* \int \cos(ax + b) \, dx = \frac{1}{a} \sin(ax + b) + C$$

$$* \sin 2x = 2 \sin x \cos x$$

$$\int \cos^2 x \, dx = \frac{1}{2} \int (1 + \cos 2x) \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \quad (8)$$

* *Observe that the antiderivatives in Formulas (3) and (4) involve both sines and cosines, whereas those in (7) and (8) involve sines alone. However, the apparent discrepancy is easy to resolve by using the identity*

$$\sin 2x = 2 \sin x \cos x$$

to rewrite (7) and (8) in forms (3) and (4), or conversely.

* *In the case where $n=3$, the reduction formulas for integrating $\sin^3 x$ and $\cos^3 x$ yield*

$$\int \sin^3 x \, dx = -\frac{1}{3}\sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx \quad (9)$$

$$= -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$$

$$\int \cos^3 x \, dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3} \int \cos x \, dx \quad (10)$$

$$= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$$

* If desired , Formula (9) can be expressed in terms of cosines alone by using the identity $\sin^2 x = 1 - \cos^2 x$, and Formula (10) can be expressed in terms of sines alone by using the identity $\cos^2 x = 1 - \sin^2 x$.

Remember that :

$$* \cos^2 x + \sin^2 x = 1$$

$$\begin{aligned}\int \sin^3 x \, dx &= -\frac{1}{3}(1 - \cos^2 x) \cos x - \frac{2}{3} \cos x + C \\ &= -\frac{1}{3} \cos x + \frac{1}{3} \cos^3 x - \frac{2}{3} \cos x + C\end{aligned}$$

$$\boxed{\int \sin^3 x \, dx = \frac{1}{3} \cos^3 x - \cos x + C} \quad (11)$$

$$\begin{aligned}\int \cos^3 x \, dx &= \frac{1}{3}(1 - \sin^2 x) \sin x + \frac{2}{3} \sin x + C \\ &= \frac{1}{3} \sin x - \frac{1}{3} \sin^3 x + \frac{2}{3} \sin x + C\end{aligned}$$

$$\boxed{\int \cos^3 x \, dx = \sin x - \frac{1}{3} \sin^3 x + C} \quad (12)$$

* We leave it as an exercise to obtain the following formulas by first applying the reduction formulas , and then using appropriate trigonometric identities.

$$\boxed{\int \sin^4 x \, dx = \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C} \quad (13)$$

* To prove Formulas (13)

$$\int \sin^4 x \, dx = \int [\sin^2 x]^2 \, dx$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1$$

$$= \int \left[\frac{1}{2}(1 - \cos 2x) \right]^2 dx$$

Remember that :

$$* (a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$= \frac{1}{4} \int [1 - 2 \cos 2x + \cos^2 2x] dx$$

Remember that :

$$* \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$= \frac{1}{4} \int \left[1 - 2 \cos 2x + \frac{1}{2}(1 + \cos 4x) \right] dx$$

$$= \frac{1}{4} \int \left[\frac{3}{2} - 2 \cos 2x + \frac{1}{2} \cos 4x \right] dx$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$* \int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$$

$$= \frac{1}{4} \left[\frac{3}{2}x - \sin 2x + \frac{1}{8} \sin 4x \right] + C$$

$$\boxed{\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C}$$

* Similarly, the following Formula

$$\boxed{\int \cos^4 x dx = \frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C} \quad (14)$$

Remember that :

$$* \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Example 1 :

Find the volume V of the solid that is obtained when the region under the curve $y = \sin^2 x$ over the interval $[0, \pi]$ is revolved about the x -axis.

Solution

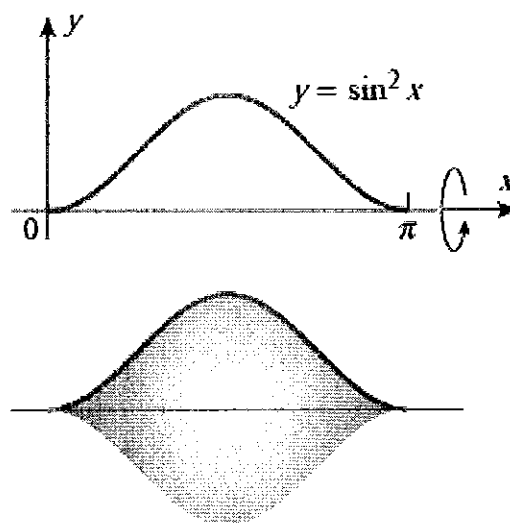
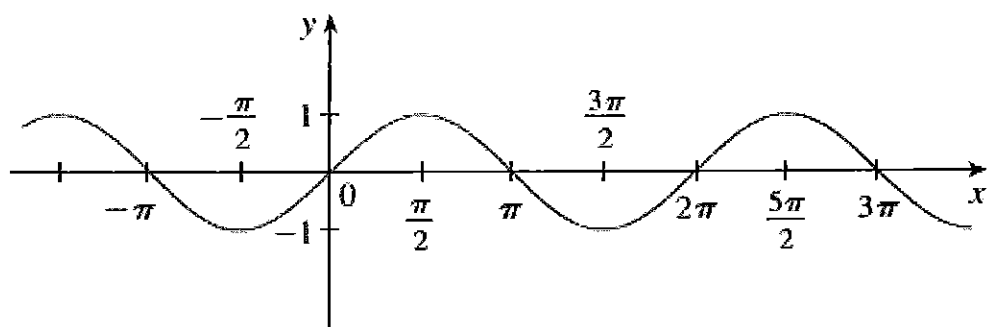


Figure 7.3.1

Remember that :

$$y = \sin x$$



* Using the method of disks , Formula (5) of Section 6.2 , and Formula (13) above

$$V = \int_a^b \pi [f(x)]^2 dx$$

$$= \int_0^{\pi} \pi [\sin^2 x]^2 dx = \pi \int_0^{\pi} \sin^4 x dx \Rightarrow \text{use Formula (13)}$$

Remember that :

$$* \int \sin^4 x dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

$$= \pi \left[\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x \right]_0^{\pi}$$

$$= \pi \left[\left(\frac{3}{8}\pi - \frac{1}{4}\sin 2\pi + \frac{1}{32}\sin 4\pi \right) - \left(\frac{3}{8}(0) - \frac{1}{4}\sin 2(0) + \frac{1}{32}\sin 4(0) \right) \right]$$

Remember that :

$$* \sin n\pi = 0 , n \text{ is an integer}$$

$$= \pi \left(\frac{3}{8}\pi \right) = \boxed{\frac{3}{8}\pi^2}.$$

Integrating Products of Sines and Cosines :

* If m and n are positive integers , then the integral

$$\int \sin^m x \cos^n x dx$$

can be evaluated by one of the three procedures stated in Table 7.3.1 , depending on whether m and n are odd or even.

Table 7.3.1

INTEGRATING PRODUCTS OF SINES AND COSINES

$\int \sin^m x \cos^n x dx$	PROCEDURE	RELEVANT IDENTITIES
n odd	• Split off a factor of $\cos x$. • Apply the relevant identity. • Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m odd	• Split off a factor of $\sin x$. • Apply the relevant identity. • Make the substitution $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
$\begin{cases} m \text{ even} \\ n \text{ even} \end{cases}$	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\begin{cases} \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{cases}$

Example 2 :

Evaluate

(a) $\int \sin^4 x \cos^5 x dx$

(b) $\int \sin^4 x \cos^4 x dx$

Solution

(a) Since $n = 5$, exponent of $\cos x$ is odd , so we will follow the first procedure in Table 7.3.1 :

$$\begin{aligned} \int \sin^4 x \cos^5 x dx &= \int \sin^4 x \cos^4 x (\cos x) dx \\ &= \int \sin^4 x (\cos^2 x)^2 (\cos x) dx \end{aligned}$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1$$

$$= \int \sin^4 x (1 - \sin^2 x)^2 (\cos x) dx$$

* We can then evaluate the integral by substituting

$$\boxed{u = \sin x}$$

$$du = \cos x dx$$

* Therefore ,

$$\int \sin^4 x \cos^5 x dx = \int u^4 (1 - u^2)^2 du$$

Remember that :

$$* (a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$= \int u^4 (1 - 2u^2 + u^4) du$$

$$= \int (u^4 - 2u^6 + u^8) du$$

Remember that :

$$* \int u^r dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{u^5}{5} - 2 \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \boxed{\frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + C}.$$

(b) Since $m = n = 4$, both exponents of $\sin x$ and $\cos x$ are even , so we will follow the third procedure in Table 7.3.1 :

$$\int \sin^4 x \cos^4 x dx = \int (\sin^2 x)^2 (\cos^2 x)^2 dx$$

Remember that :

$$* \sin^2 x = \frac{1}{2}(1 - \cos 2x) , \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$= \int \left(\frac{1}{2} [1 - \cos 2x] \right)^2 \left(\frac{1}{2} [1 + \cos 2x] \right)^2 dx$$

Remember that :

$$* (a-b)(a+b) = a^2 - b^2$$

$$= \frac{1}{16} \int (1 - \cos^2 2x)^2 dx$$

$$= \frac{1}{16} \int \sin^4 2x dx$$

* We can then evaluate the integral by substituting

$$\boxed{u = 2x}$$

$$du = 2 dx \quad \text{or} \quad \frac{1}{2} du = dx$$

Therefore ,

$$\int \sin^4 x \cos^4 x dx$$

$$= \frac{1}{32} \int \sin^4 u du \Rightarrow \text{use Formula (13)}$$

Remember that :

$$* \int \sin^4 x dx = \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

$$= \frac{1}{32} \left(\frac{3}{8} u - \frac{1}{4} \sin 2u + \frac{1}{32} \sin 4u \right) + C$$

$$= \boxed{\frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x + C} .$$

* *Integrals of the form*

$$\int \sin mx \cos nx \, dx, \quad \int \sin mx \sin nx \, dx, \quad \int \cos mx \cos nx \, dx \quad (15)$$

can be found by using the trigonometric identities

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)] \quad (16)$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \quad (17)$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] \quad (18)$$

to express the integrand as a sum or difference of sines and cosines.

Example 3 :

Evaluate $\int \sin 7x \cos 3x \, dx$.

Solution

* *Using (16) yields*

$$\int \sin 7x \cos 3x \, dx = \frac{1}{2} \int (\sin 4x + \sin 10x) \, dx$$

Remember that :

$$* \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$* \int \sin(ax + b) \, dx = -\frac{1}{a} \cos(ax + b) + C$$

$$= \frac{1}{2} \left(-\frac{1}{4} \cos 4x - \frac{1}{10} \cos 10x \right) + C$$

$$= \boxed{-\frac{1}{8} \cos 4x - \frac{1}{20} \cos 10x + C}.$$

Integrating Powers of Tangent and Secant :

* The procedures for integrating powers of tangent and secant closely parallel those for sine and cosine. The idea is to use the following reduction formulas to reduce the exponent in the integrand until the resulting integral can be evaluated :

$$\boxed{\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx} \quad (19)$$

* To derive reduction formula (19) we begin by writing $\tan^n x$ as $\tan^{n-2} x \cdot \tan^2 x$ yields

$$\int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1 \Rightarrow 1 + \tan^2 x = \sec^2 x$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \int (\tan x)^{n-2} \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

Remember that :

$$* \int [g(x)]^n g'(x) \, dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\boxed{\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx}.$$

$$\boxed{\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx} \quad (20)$$

* To derive reduction formula (20) we begin by writing $\sec^n x$ as $\sec^{n-2} x \cdot \sec^2 x$ and letting

$$u = \sec^{n-2} x$$

$$dv = \sec^2 x \, dx$$

$$du = (n-2) \sec^{n-2} x (\tan x) \, dx$$

$$v = \int \sec^2 x \, dx = \tan x$$

Remember that :

$$* \int \sec^2 x \, dx = \tan x + C$$

Thus , from (3)

Remember that :

$$* \int u \, dv = uv - \int v \, du$$

$$\begin{aligned} \int \sec^n x \, dx &= \int \underbrace{\sec^{n-2} x}_u \underbrace{\sec^2 x \, dx}_{dv} \\ &= \underbrace{\sec^{n-2} x}_u \underbrace{(\tan x)}_v - \int \underbrace{(\tan x)}_v \underbrace{(n-2) \sec^{n-2} x \tan x \, dx}_{du} \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx \end{aligned}$$

Remember that :

$$* 1 + \tan^2 x = \sec^2 x$$

$$\begin{aligned} &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \\ &= \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx \end{aligned}$$

$$(n-1) \int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx .$$

* In the case where n is odd, the exponent can be reduced to 1, leaving us with the problem of integrating $\tan x$ or $\sec x$. These integrals are given by

$$\int \tan x \, dx = \ln |\sec x| + C \quad (21)$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C \quad (22)$$

* Formula (21) can be obtained by writing

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

Substituting $u = \cos x$, then

$$du = -\sin x \, dx \quad \text{or} \quad -du = \sin x \, dx$$

Thus,

$$\int \tan x \, dx = -\int \frac{1}{u} \, du$$

Remember that :

$$* \int \frac{1}{u} \, du = \ln |u| + C$$

$$= -\ln |u| + C = -\ln |\cos x| + C = \ln \frac{1}{|\cos x|} + C$$

Remember that :

$$* \ln x^r = r \ln x, \quad \frac{1}{\cos x} = \sec x$$

$$= \ln |\sec x| + C .$$

Note :

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = - \int \frac{-\sin x}{\cos x} \, dx$$

Remember that :

$$* \int \frac{g'(x)}{g(x)} \, du = \ln |g(x)| + C$$

$$= - \ln |\cos x| + C = \ln \frac{1}{|\cos x|} + C$$

Remember that :

$$* \ln x^r = r \ln x, \quad \frac{1}{\cos x} = \sec x$$

$$= \boxed{\ln |\sec x| + C}.$$

* To obtain Formula (22) we write

$$\begin{aligned} \int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \end{aligned}$$

Substituting $\boxed{u = \sec x + \tan x}$, then

$$du = (\sec^2 x + \sec x \tan x) \, dx$$

Thus,

$$\int \sec x \, dx = \int \frac{1}{u} \, du$$

Remember that :

$$* \int \frac{1}{u} \, du = \ln |u| + C$$

$$= \ln |u| + C = \boxed{\ln |\sec x + \tan x| + C}.$$

Note :

$$\begin{aligned}\int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx\end{aligned}$$

Remember that :

$$* \int \frac{g'(x)}{g(x)} du = \ln |g(x)| + C$$

$$= \boxed{\ln |\sec x + \tan x| + C}.$$

* The following basic integrals occur frequently and are worth noting :

$$\boxed{\int \tan^2 x \, dx = \tan x - x + C} \quad (23)$$

$$\boxed{\int \sec^2 x \, dx = \tan x + C} \quad (24)$$

* Formula (24) is already known to us, since the derivative of $\tan x$ is $\sec^2 x$. Formula (23) can be obtained by applying reduction formula (19) with $n = 2$ or, alternatively, by using the identity

$$1 + \tan^2 x = \sec^2 x$$

to write

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \boxed{\tan x - x + C}.$$

* The formula

$$\boxed{\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \ln |\sec x| + C} \quad (25)$$

can be deduced from (21) and reduction formula (19) as follows :

$$\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \int \tan x \, dx$$

Remember that :

$$* \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

$$* \int \tan x \, dx = \ln |\sec x| + C$$

$$= \boxed{\frac{1}{2} \tan^2 x - \ln |\sec x| + C}.$$

Another Method :

* We can evaluate the integral $\int \tan^3 x \, dx$ as follows :

$$\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1 \Rightarrow 1 + \tan^2 x = \sec^2 x$$

$$= \int \tan x (\sec^2 x - 1) \, dx$$

$$= \int (\tan x)^1 \sec^2 x \, dx - \int \tan x \, dx$$

Remember that :

$$* \int [g(x)]^n g'(x) \, dx = \frac{[g(x)]^{n+1}}{n+1} + C, n \neq -1$$

$$* \int \tan x \, dx = \ln |\sec x| + C$$

$$\boxed{\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \ln |\sec x| + C}.$$

* *The formula*

$$\boxed{\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C} \quad (26)$$

can be deduced from (22) and reduction formula (20) as follows :

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx$$

Remember that :

$$* \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

$$* \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$= \boxed{\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C}.$$

Another Method :

* We can evaluate the integral $\int \sec^3 x \, dx$ by integration by parts as follows :

$$\int \sec^3 x \, dx = \int \underbrace{\sec x}_u \underbrace{\sec^2 x \, dx}_{dv}, \text{ let}$$

$$u = \sec x \quad dv = \sec^2 x \, dx$$

$$du = \sec x \tan x \, dx \quad v = \tan x$$

Remember that :

$$* \int \sec^2 x \, dx = \tan x + C$$

Thus ,

Remember that :

$$* \int u \, dv = uv - \int v \, du$$

$$\begin{aligned}
\int \sec^3 x \, dx &= \underbrace{\sec x}_u \underbrace{\tan x}_v - \int \underbrace{\tan x}_v \underbrace{\sec x \tan x \, dx}_{du} \\
&= \sec x \tan x - \int \sec x \tan^2 x \, dx \\
&= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\
2 \int \sec^3 x \, dx &= \sec x \tan x + \int \sec x \, dx \\
\int \sec^3 x \, dx &= \frac{1}{2} \left[\sec x \tan x + \int \sec x \, dx \right]
\end{aligned}$$

Remember that :

$$* \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\boxed{\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C}.$$

Integrating Products of Tangents and Secants :

* If m and n are positive integers , then the integral

$$\int \tan^m x \sec^n x dx$$

can be evaluated by one of the three procedures stated in Table 7.3.2 , depending on whether m and n are odd or even.

Table 7.3.2

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

$\int \tan^m x \sec^n x dx$	PROCEDURE	RELEVANT IDENTITIES
n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
$\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

Example 4 :

Evaluate

(a) $\int \tan^2 x \sec^4 x dx$

(b) $\int \tan^3 x \sec^3 x dx$

(a) $\int \tan^2 x \sec x dx$

Solution

(a) Since $n = 4$, exponent of $\sec x$ is even , so we will follow the first procedure in Table 7.3.2 :

$$\int \tan^2 x \sec^4 x dx = \int \tan^2 x \sec^2 x (\sec^2 x) dx$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1$$

$$* 1 + \tan^2 x = \sec^2 x$$

$$= \int \tan^2 x (\tan^2 x + 1) (\sec^2 x) dx$$

* We can then evaluate the integral by substituting

$$\boxed{u = \tan x}$$

$$du = \sec^2 x dx$$

* Therefore ,

$$\begin{aligned} \int \tan^2 x \sec^4 x dx &= \int u^2 (u^2 + 1) du \\ &= \int (u^4 + u^2) du \end{aligned}$$

Remember that :

$$* \int u^r dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{u^5}{5} + \frac{u^3}{3} + C$$

$$= \boxed{\frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C} .$$

(b) Since $m = 3$, exponent of $\tan x$ is odd , so we will follow the second procedure in Table 7.3.2 :

$$\int \tan^3 x \sec^3 x dx = \int \tan^2 x \sec^2 x (\sec x \tan x) dx$$

Remember that :

$$* 1 + \tan^2 x = \sec^2 x$$

$$= \int (\sec^2 x - 1) \sec^2 x (\sec x \tan x) dx$$

* We can then evaluate the integral by substituting

$$\boxed{u = \sec x}$$

$$du = \sec x \tan x \, dx$$

* Therefore ,

$$\begin{aligned} \int \tan^3 x \sec^3 x \, dx &= \int (u^2 - 1) u^2 \, du \\ &= \int (u^4 - u^2) \, du \end{aligned}$$

Remember that :

$$* \int u^r \, dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \boxed{\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C} .$$

(c) Since $m = 2$, exponent of $\tan x$ is even , and $n = 1$, exponent of $\sec x$ is odd , so we will follow the third procedure in Table 7.3.2 :

$$\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx$$

Remember that :

$$* 1 + \tan^2 x = \sec^2 x$$

$$= \int \sec^3 x \, dx - \int \sec x \, dx$$

* Using Formula (22) and Formula (26)

Remember that :

$$* \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$* \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$

$$\begin{aligned} \int \tan^2 x \sec x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| \\ &\quad - \ln |\sec x + \tan x| + C \\ &= \boxed{\frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C} . \end{aligned}$$

**An Alternative method for Integrating Powers of Sine , Cosine ,
Tangent , and Secant :** **Page (505)**

* The methods in Tables 7.3.1 and 7.3.2 can sometimes be applied if $m = 0$ or $n = 0$ to integrate positive integer powers of sine , cosine , tangent , and secant without reduction formulas. For example , instead of using the reduction formula to integrate $\sin^3 x$, we can apply the second procedure in Table 7.3.1:

$$\int \sin^3 x \, dx = \int \sin^2 x (\sin x) \, dx$$

Remember that :

$$* \cos^2 x + \sin^2 x = 1$$

$$= \int (1 - \cos^2 x) (\sin x) \, dx$$

* We can then evaluate the integral by substituting

$$\boxed{u = \cos x}$$

$$du = -\sin x \, dx \quad \text{or} \quad -du = \sin x \, dx$$

* Therefore ,

$$\int \sin^3 x \, dx = - \int (1 - u^2) \, du$$

Remember that :

$$* \int u^r dx = \frac{u^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$= \frac{u^3}{3} - u + C = \boxed{\frac{1}{3} \cos^3 x - \cos x + C}.$$

which agrees with (11).

Note :

* With the aid of the identity

$$1 + \cot^2 x = \csc^2 x$$

the techniques in Table 7.3.2 can be adapted to evaluate integrals of the form

$$\int \cot^m x \csc^n x dx$$

it is also possible to derive reduction formulas for powers of $\cot$ and $\csc$ that are analogous to Formula (19) and (20).

EXERCISE SET 7.3:

(Home Work)

Evaluate the integral.

9. $\int \sin^2 t \cos^3 t dt$

10. $\int \sin^3 x \cos^2 x dx$

11. $\int \sin^2 x \cos^2 x dx$

14. $\int \sin 3\theta \cos 2\theta d\theta$

32. $\int \tan^4 x \sec^4 x dx$

33. $\int \sec^5 x \tan^3 x dx$

7.5 INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS :

Partial Fractions :

* In algebra , one learns to combine two or more fractions into a single fraction by finding a common denominator. For example ,

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{2(x+1)+3(x-4)}{(x-4)(x+1)} = \frac{5x-10}{x^2-3x-4} \quad (1)$$

Remember that :

$$* \frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

* However , for purposes of integration , the left side of (1) is preferable to the right side since each of the terms is easy to integrate :

$$\int \frac{5x-10}{x^2-3x-4} dx = \int \frac{2}{x-4} dx + \int \frac{3}{x+1} dx$$

Remember that :

$$* \int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C$$

$$= \boxed{2 \ln|x-4| + 3 \ln|x+1| + C} .$$

$$\frac{5x-10}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1} \quad (2)$$

* One way to find the constants A and B is to multiply (2) through by $(x-4)(x+1)$ to clear fractions. This yields

$$5x-10 = A(x+1) + B(x-4) \quad (3)$$

This relationship holds for all x , so it holds in particular if $x=4$ or $x=-1$. Substituting $x=4$ in (3) makes the second term on the right drop out and yields the equation $10=5A$ or $A=2$; and substituting $x=-1$ in (3) makes the first term on the right drop out and yields the equation $-15=-5B$ or $B=3$. Substituting these values in (2) we obtain

$$\frac{5x-10}{(x-4)(x+1)} = \frac{2}{x-4} + \frac{3}{x+1} \quad (4)$$

which agrees with (1).

* A second method for finding the constants A and B is to multiply out the right side of (3) and collect like powers of x to obtain

$$5x-10 = (A+B)x + (A-4B)$$

Since the polynomials on the two sides are identical, their corresponding coefficients must be the same. Equating the corresponding coefficients on the two sides yields the following system of equations in the unknowns A and B :

$$5 = A + B$$

$$-10 = A - 4B$$

Solving this system yields $A=2$ and $B=3$ as before.

Finding the Forms of a Partial Fraction Decomposition :

* The first step in finding the form of the partial fraction decomposition of a proper rational function $P(x)/Q(x)$ is to factor $Q(x)$ completely into linear and irreducible quadratic factors, and then collect all repeated factors so that is expressed as a product of distinct factors of the form

$$(ax + b)^m \quad \text{and} \quad (ax^2 + bx + c)^m$$

* From these factors we can determine the form of the partial fraction decomposition using two rules that we will now discuss.

Linear Factors :

* If all of the factors of $Q(x)$ are linear, then the partial fraction decomposition of $P(x)/Q(x)$ can be determined by using the following rule:

Linear Factor Rule :

For each factor of the form $(ax + b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_m}{(ax + b)^m}$$

where $A_1, A_2, \dots, A_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 1:

Evaluate $\int \frac{dx}{x^2 + x - 2}$.

Solution

* *The integrand is a proper rational function that can be written as*

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x-1)(x+2)}$$

Remember that:

* *If degree of numerator < degree of denominator
 $\Rightarrow$ proper rational function*

* *The factors $x-1$ and $x+2$ are both linear and appear to the first power, so each contributes one term to the partial fraction decomposition by the linear factor rule. Thus, the decomposition has the form*

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \quad (5)$$

where A and B are constants to be determined.

* *Multiplying this expression through by $(x-1)(x+2)$ yields*

$$1 = A(x+2) + B(x-1) \quad (6)$$

* *As discussed earlier, there are two methods for finding A and B : we can substitute values of x that are chosen to make terms on the right drop out, or we can multiply out on the right and equate corresponding coefficients on the two sides to obtain a system of equations that can be solved for A and B .*

* *We will use the first approach, setting $\boxed{x=1}$ makes the second term in (6) drop out and yields $1 = 3A$ or $A = \boxed{\frac{1}{3}}$;*

and setting $x = -2$ makes the first term in (6) drop out and yields $1 = -3B$ or $B = -\frac{1}{3}$.

* If we use the second approach, equating the corresponding coefficients on the two sides of (6) yields:

$$1 = A + B$$

$$0 = 2A - B$$

Solving this system yields $A = \frac{1}{3}$ and $B = -\frac{1}{3}$ as before.

* Substituting these values in (5) yields the partial fraction decomposition

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}}{x+2}$$

* The integration can now be completed as follows:

$$\int \frac{dx}{(x-1)(x+2)} = \frac{1}{3} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{dx}{x+2}$$

Remember that :

$$* \int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2| + C$$

$$= \frac{1}{3} [\ln|x-1| - \ln|x+2|] + C$$

Remember that :

$$* \ln a - \ln b = \ln\left(\frac{a}{b}\right), \quad a, b > 0$$

$$= \boxed{\frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C}.$$

* If the factors of $Q(x)$ are linear and none are repeated, as in the last example, then the recommended method for finding the constants in the partial fraction decomposition is to substitute appropriate values of x to make terms drop out. However, if some of the linear factors are repeated, then it will be possible to find all of the constants in this way. In this case the recommended procedure is to find as many constants as possible by substitution and then find the rest by equating coefficients. This is illustrated in the next example.

Example 2 :

Evaluate $\int \frac{2x+4}{x^3-2x^2} dx$.

Solution

* The integrand is a proper rational function that can be written as

Remember that :

* If degree of numerator < degree of denominator
 $\Rightarrow$ proper rational function

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

* Although x^2 is a quadratic factor, it is not irreducible since $x^2 = x \cdot x$. Thus, by the linear factor rule, x^2 introduces two terms (since $m = 2$) of the form

$$\frac{A}{x} + \frac{B}{x^2}$$

and the factor $x - 2$ introduces one term (since $m = 1$) of the form

$$\frac{C}{x - 2}$$

so the partial fraction decomposition is

$$\frac{2x + 4}{x^2(x - 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 2} \quad (7)$$

* Multiplying by $x^2(x - 2)$ yields

$$2x + 4 = Ax(x - 2) + B(x - 2) + Cx^2 \quad (8)$$

which, after multiplying out and collecting like powers of x , becomes

$$2x + 4 = (A + C)x^2 + (-2A + B)x - 2B \quad (9)$$

* Setting $x = 0$ in (8) makes the first and third terms drop out and yields $B = -2$, and setting $x = 2$ in (8) makes the first and second terms drop out and yields $C = 2$. However, there is no substitution in (8) that produces A directly, so we look to equation (9) to find this value. This can be done by equating the coefficients of x^2 on the two sides to obtain

$$A + C = 0 \quad \text{or} \quad A = -C = -2$$

* Substituting the values $A = -2$, $B = -2$, and $C = 2$ in (7) yield the partial fraction decomposition

$$\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

* Thus,

$$\int \frac{2x+4}{x^2(x-2)} dx = -2 \int \frac{1}{x} dx - 2 \int \frac{1}{x^2} dx + 2 \int \frac{1}{x-2} dx$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$* \int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C$$

$$= -2 \ln|x| + \frac{2}{x} + 2 \ln|x-2| + C$$

$$= 2 \left[\ln|x-2| - \ln|x| \right] + \frac{2}{x} + C$$

Remember that :

$$* \ln a - \ln b = \ln\left(\frac{a}{b}\right), \quad a, b > 0$$

$$= \boxed{2 \ln \left| \frac{x-2}{x} \right| + \frac{2}{x} + C}.$$

Quadratic Factors :

* If all of the factors of $Q(x)$ are irreducible quadratics, then the partial fraction decomposition of $P(x)/Q(x)$ can be determined from the following rule:

Quadratic Factor Rule :

For each factor of the form $(ax^2 + bx + c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

where $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 3 :

Evaluate $\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} dx$.

Solution

* The denominator in the integrand can be factored by grouping:

$$\begin{aligned} 3x^3 - x^2 + 3x - 1 &= x^2(3x - 1) + (3x - 1) \\ &= (3x - 1)(x^2 + 1) \end{aligned}$$

By the linear factor rule, the factor $3x - 1$ introduces one term, namely

$$\frac{A}{3x-1}$$

and by the quadratic factor, the factor $x^2 + 1$ introduces one term, namely

$$\frac{Bx + C}{x^2 + 1}$$

Thus, the partial fraction decomposition is

$$\frac{x^2 + x - 2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1} \quad (10)$$

* Multiplying by $(3x-1)(x^2+1)$ yields

$$x^2 + x - 2 = A(x^2 + 1) + (Bx + C)(3x - 1) \quad (11)$$

* We could find A by substituting $x = \frac{1}{3}$ to make the last

term drop out yields $\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right) - 2 = A\left(\left(\frac{1}{3}\right)^2 + 1\right)$ so

$$A = \frac{-14}{10} = -\frac{7}{5}, \text{ and then the rest of the constants by}$$

equating corresponding coefficients. However, in the case it is just as easy to find all of the constants by equating coefficients and solving the resulting system. For this purpose we multiply out the side of (11) and collect like terms:

$$x^2 + x - 2 = (A + 3B)x^2 + (-B + 3C)x + (A - C)$$

* Equating corresponding coefficients gives

$$A + 3B = 1$$

$$-B + 3C = 1$$

$$A - C = -2$$

* From first equation $B = \frac{1}{3}(1 - A) = \frac{1}{3}\left(1 - \left(-\frac{7}{5}\right)\right) = \boxed{\frac{4}{5}}$, from
third equation $C = A + 2 = -\frac{7}{5} + 2 = \boxed{\frac{3}{5}}$.

* Second method is solving this system of three equations, subtract the third equation from the first to eliminate A . Then use the resulting equation together with the second equation to solve for B and C . Finally, determine A from the first or third equation. This yields

$$A = \boxed{-\frac{7}{5}}, \quad B = \boxed{\frac{4}{5}}, \quad C = \boxed{\frac{3}{5}}$$

* Thus, (10) becomes

$$\frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} = \frac{-\frac{7}{5}}{3x - 1} + \frac{\frac{4}{5}x + \frac{3}{5}}{x^2 + 1}$$

and

$$\begin{aligned} \int \frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} dx &= -\frac{7}{5} \int \frac{1}{3x - 1} dx + \frac{2}{5} \int \frac{x}{x^2 + 1} dx \\ &\quad + \frac{2}{5} \int \frac{1}{x^2 + 1} dx \\ &= -\frac{7}{5} \frac{1}{3} \int \frac{3}{3x - 1} dx + \frac{1}{5} \int \frac{2x}{x^2 + 1} dx + \frac{2}{5} \int \frac{1}{x^2 + 1} dx \end{aligned}$$

Remember that :

$$* \int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C$$

$$* \int \frac{1}{a^2 + x^2} du = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$= \boxed{-\frac{7}{15} \ln|3x-1| + \frac{2}{5} \ln(x^2+1) + \frac{3}{5} \tan^{-1} x + C}.$$

Integrating Improper Rational Functions :

* Although the method of partial fractions only applies to proper rational functions, an improper rational function can be integrated by performing a long division and expressing the function as the quotient plus the remainder over the divisor. The remainder over the divisor will be a proper rational function, which can then be decomposed into fractions. This idea is illustrated in the following example.

Example 5 :

Evaluate $\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx$.

Solution

* The integrand is an improper rational function since the numerator has degree 4 and the denominator has degree 2. Thus, we first perform the long division

Remember that :

* If degree of numerator $\geq$ degree of denominator
 $\Rightarrow$ improper rational function

$$\begin{array}{r}
 3x^2 + 1 \\
 x^2 + x - 2 \overline{) 3x^4 + 3x^3 - 5x^2 + x - 1} \\
 \underline{3x^4 + 3x^3 - 6x^2} \\
 x^2 + x - 1 \\
 \underline{x^2 + x - 2} \\
 1
 \end{array}$$

* It follows that the integrand can be expressed as

$$\frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} = (3x^2 + 1) + \frac{1}{x^2 + x - 2}$$

and hence

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx = \int (3x^2 + 1) dx + \int \frac{1}{x^2 + x - 2} dx$$

* The second integrand on the right now involves a proper rational function and can thus be evaluated by a partial fraction decomposition. Using the result of Example 1 we obtain

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx = \boxed{x^3 + x + \frac{1}{3} \ln \left| \frac{x-1}{x+1} \right| + C}.$$

Remember that :

$$* \int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

EXERCISE SET 7.5:**(Home Work)*****Evaluate the integral.***

$$1. \int \frac{3x-1}{(x-3)(x+4)} dx$$

$$6. \int \frac{3x}{(x-1)(x^2+6)} dx$$

$$9. \int \frac{dx}{x^2-3x-4}$$

$$23. \int \frac{2x^2+3}{x(x-1)^2} dx$$